# Elementary Logic (PrepCamp)

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**Slides and exercises:**

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# Statements

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**Example (A bad joke)**

**Q:** How many months have 30 days?

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- Objective

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# Statements

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**Example (A bad joke)**

**Q:** How many months have 30 days?

**A:** 11, some of them have even more!

:-(

- Objective

**Example**

**Good:** 3 is greater than 4

**Bad:** 3 is nicer than 4

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# Statements

- Mathematical: "*Three is greater than four*" (or "$3 > 4$")
- ...or not: "*I am 26 years old*"
- **Key point:** staments can be **true** or **false**

# Logical operations

- We can combine statements to make new ones
- Negation (**not**), conjunction (**and**), disjunction (**or**)

# Negation (**not**)

If $A$ is a statement, the statement "not $A$" (in symbols: $\neg A$) is **true** when $A$ is **false**, and it is **false** when $A$ is **true**.

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## Example

$\neg(3 > 4)$ is equivalent to $3 \leq 4$

"3 *is* not *greater than* 4" is equivalent to "3 *is less or equal than* 4"

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# Conjunction (**and**)

The statement "$A$ and $B$" (in symbols: $A \wedge B$) is **true** when both $A$ and $B$ are **true**, and it is **false** if at *at least* one of them is **false**.

# Conjunction (**and**)

The statement "$A$ and $B$" (in symbols: $A \wedge B$) is **true** when both $A$ and $B$ are **true**, and it is **false** if at *at least* one of them is **false**.

**Example**

"$(3 < 4) \wedge$ (5 is an odd number)" is **true**

**Example**

"(Today is Monday) $\wedge$ (we are in France)" is **false**

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# Disjunction (**or**)

The statement "$A$ or $B$" (in symbols: $A \vee B$) is **true** when at least one of $A$ and $B$ is **true**, and it is **false** if both of them are **false**.

# Disjunction (**or**)

The statement "$A$ or $B$" (in symbols: $A \lor B$) is **true** when at least one of $A$ and $B$ is **true**, and it is **false** if both of them are **false**.

**Example**

"$(3 = 4) \lor$ (5 is an even number)" is **false**

**Example**

"(Today is Monday) $\lor$ (we are in Luxembourg)" is **true**

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# Logical operations

- **Important:** $\vee$ is always *inclusive*:

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**Example (Another bad joke)**

Waiter: "Would you like cheese or dessert?"
Mathematician: "Yes."

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**Example (Another bad joke)**

Waiter: "Would you like cheese or dessert?"
Mathematician: "Yes."

- $\neg$ has precedence over $\wedge$ and $\vee$:

$$\neg A \wedge B \text{ means } (\neg A) \wedge B, \qquad \neg A \vee B \text{ means } (\neg A) \vee B$$

(or just use parenthesis)

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# Properties

If $A$, $B$ and $C$ are statements:

| | | |
|---|---|---|
| $A \wedge B = B \wedge A$ | $A \vee B = B \vee A$ | **commutativity** |
| $A \wedge (B \wedge C) = (A \wedge B) \wedge C$ | $A \vee (B \vee C) = (A \vee B) \vee C$ | **associativity** |
| $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ | | **distributivity** |
| $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ | | **distributivity*** |
| $\neg(\neg A) = A$ | | **double negation** |
| $A \wedge \textbf{true} = A$ | $A \wedge \textbf{false} = \textbf{false}$ | |
| $A \vee \textbf{true} = \textbf{true}$ | $A \vee \textbf{false} = A$ | |
| $(\neg A) \wedge A = \textbf{false}$ | $(\neg A) \vee A = \textbf{true}$ | |
| $\neg(A \wedge B) = (\neg A) \vee (\neg B)$ | $\neg(A \vee B) = (\neg A) \wedge (\neg B)$ | **De Morgan's laws** |

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# Boolean algebra

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- We have a set $\{0, 1\}$ with some operations $(\wedge, \vee, \neg)$
- This is called a **Boolean algebra**

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# Truth tables

A compact way of describing an operator, or a composition of operators

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Example:

| $A$ | $B$ | $\neg A$ | $A \wedge B$ | $A \vee B$ | $(A \vee B) \wedge (\neg A)$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 |

# Truth tables

We can check that two statements are equivalent with truth tables

| $A$ | $B$ | $\neg(A \wedge B)$ | $(\neg A) \vee (\neg B)$ |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |

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**Example**

*"If it rains, I will bring an umbrella"*

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# Implication

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**Example**

*"If it rains, I will bring an umbrella"*
(It rains) $\implies$ (I will bring an umbrella)

**Example**

*"If my grandpa had wheels, he would be a bike"*
(My grandpa has wheels) $\implies$ (My grandpa is a bike)

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# Implication

- It is a logical operation: "$A \implies B$" means "$B \vee (\neg A)$"

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| $A$ | $B$ | $A \implies B$ | |
|---|---|---|---|
| 0 | 0 | 1 | No rain, I don't bring an umbrella |
| 0 | 1 | 1 | No rain, I bring an umbrella anyway |
| 1 | 0 | 0 | It rains, I don't bring an umbrella |
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## Remark

"**false** $\implies A$" is always **true**, whatever $A$ is (*ex falso quodlibet*)
"$A \implies$ **true**" is always true, whatever $A$ is

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# Notation

Sometimes we use the following symbols:

- "$A \impliedby B$" is the same as "$B \implies A$"
- "$A \iff B$ is the same as "$(A \implies B) \wedge (B \implies A)$".
  It is read "$A$ is equivalent to $B$" or "$A$ if and only if $B$".

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# Contrapositive

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# Contrapositive

- The statement $(\neg B) \implies (\neg A)$ is called *contrapositive* of $A \implies B$
- It is equivalent to "$A \implies B$"
- Two proofs:
  1. Properties of logical operations
  2. Truth tables

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# End of part 1

## See you tomorrow!

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# Welcome Back!

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Let $S$ be a set and let $A(x)$ be a "variable statement" that depends on $x \in S$ (for example $S = \mathbb{N}$ and $A(x) =$"x is an even number").

# Quantifiers

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- **Universal quantifier** ($\forall$ or "for all"): "$\forall x \in S$, $A(x)$" means that if we replace $x$ with any element of $S$, $A(x)$ is always **true**.
- **Existential quantifier** ($\exists$ or "there exists"): "$\exists x \in S$, $A(x)$" means that $A(x)$ is **true** for at least one value of $x$ is $S$.

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# Quantifiers - examples

**Example**

$S =$ "the set of all cars", $A(x) =$ "$x$ is red"
$\forall x \in S,\ A(X)$ is **false**.
$\exists x \in S,\ A(X)$ is **true**.

**Example**

$S = \mathbb{N},\ A(x) = x > 5$
$\forall x \in S,\ A(x)$ is **false**.
$\exists x \in S,\ A(X)$ is **true**.

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**Today's most important fact:**

$$\neg(\forall x \in S,\ A(x)) = \quad ?$$
$$\neg(\exists x \in S,\ A(x)) = \quad ?$$

# Negation of quantifiers

**Today's most important fact:**

$$\neg(\forall x \in S,\ A(x)) = \exists x \in S,\ \neg A(x)$$
$$\neg(\exists x \in S,\ A(x)) = \quad ?$$

**Example**

$\neg$"every number is even" $=$ "there is at least one odd number"

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**Example**

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(exercise)

# Proofs

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# Proofs

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- Proofs are used to derive new statements from statements that are known to be true.
- If $A$ is known to be true and the implication $A \implies B$ is logically clear, then also $B$ must be true.
- Every mathematical theorem must be justified with a proof.

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# Example: direct proof

**Theorem**

*The sum of two even numbers is even.*

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## Proof.

1. Recall the definition: a natural number $x$ is called *even* if there is some natural number $n$ such that $x = 2n$.

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**Proof.**

1. Recall the definition: a natural number $x$ is called *even* if there is some natural number $n$ such that $x = 2n$.
2. If $x$ and $y$ are even numbers, then there are natural numbers $n$ and $m$ such that $x = 2n$ and $y = 2m$.

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# Example: direct proof

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1. Recall the definition: a natural number $x$ is called *even* if there is some natural number $n$ such that $x = 2n$.
2. If $x$ and $y$ are even numbers, then there are natural numbers $n$ and $m$ such that $x = 2n$ and $y = 2m$.
3. Then $x + y = 2n + 2m = 2(n + m)$.

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3. Then $x + y = 2n + 2m = 2(n + m)$.
4. Then $x + y$ is even.

□

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# Proofs by contradiction

Idea: I want to show $A =$ **true**. I show that the implication "$(\neg A) \implies$ **false**" is **true**. Then $\neg A =$ **false**, so $A =$ **true**.

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Idea: I want to show $A = \textbf{true}$. I show that the implication "$(\neg A) \implies$ **false**" is **true**. Then $\neg A = \textbf{false}$, so $A = \textbf{true}$.

## Definition

A natural number is called *prime* if it is different from 1 and it is only divisible by 1 and itself.

## Theorem

*There are infinitely many prime numbers.*

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# Proofs by contradiction

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1. Assume that there are only finitely many prime numbers.

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1. Assume that there are only finitely many prime numbers.
2. So there are $n$ prime numbers, for some number $n$. Call them $p_1, p_2, \ldots, p_n$.

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1. Assume that there are only finitely many prime numbers.
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3. Let $u = p_1 \times p_2 \times \cdots \times p_n + 1$.

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4. $u$ is not divisible by any of the prime numbers $p_1, \ldots, p_n$.
5. Therefore $u$ is only divisible by 1 and itself. So $u$ is prime.

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## Theorem

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1. Assume that there are only finitely many prime numbers.
2. So there are $n$ prime numbers, for some number $n$. Call them $p_1, p_2, \ldots, p_n$.
3. Let $u = p_1 \times p_2 \times \cdots \times p_n + 1$.
4. $u$ is not divisible by any of the prime numbers $p_1, \ldots, p_n$.
5. Therefore $u$ is only divisible by 1 and itself. So $u$ is prime.
6. So $p_1, \ldots, p_n$ are not the only prime numbers.

$\square$

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# Proofs by induction

If I want to prove $\forall n \in \mathbb{N},\ A(n)$:

1. Prove $A(0)$ (*base step*)
2. Prove $\forall n \in \mathbb{N},\ (A(n) \implies A(n+1))$ (*inductive step*)

# Proofs by induction

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**Theorem (Sum of natural numbers)**

$$\forall n \in \mathbb{N}, \quad 0 + 1 + \cdots + n = \frac{n(n+1)}{2}$$

## Proof.

1. Base case: $0 = 0$.

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1. Base case: $0 = 0$.
2. Let $n$ be any natural number.
   If $A(n) =$ **false**, then $A(n) \implies A(n+1)$ is **true**.

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   If $A(n) =$ **true**, we have to show that $A(n+1) =$ **true**.

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**Proof.**

1. Base case: $0 = 0$.
2. Let $n$ be any natural number.
   If $A(n) = \textbf{false}$, then $A(n) \implies A(n+1)$ is **true**.
   If $A(n) = \textbf{true}$, we have to show that $A(n+1) = \textbf{true}$.

$$\begin{aligned} 0 + \cdots + (n+1) &= (0 + \cdots + n) + (n+1) = \\ &= \tfrac{n(n+1)}{2} + (n+1) = \qquad \text{since } A(n) = \textbf{true} \\ &= \tfrac{n^2+n+2n+2}{2} \\ &= \tfrac{(n+1)(n+2)}{2} \end{aligned}$$

so $A(n+1) = \textbf{true}$

$\square$

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