Elementary Logic (PrepCamp)

Sebastiano Tronto

uni.lu

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- **These slides:**
- **Exercises:**
- **Contact:**

`sebastiano.tronto@uni.lu`

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Statements

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- Unambiguous

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Example (A bad joke)

Q: How many months have 30 days?

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- Unambiguous

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Q: How many months have 30 days?

A: 11, some of them have even more!

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- Objective

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- Unambiguous

Example (A bad joke)

Q: How many months have 30 days?

A: 11, some of them have even more!

:-(

- Objective

Example

Good: 3 is greater than 4

Bad: 3 is nicer than 4

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Statements

- Mathematical: "*Three is greater than four*" (or "$3 > 4$")
- ...or not: "*I am 26 years old*"
- **Key point:** staments can be **true** or **false**

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Logical operations

- We can combine statements to make new ones
- Negation (**not**), conjunction (**and**), disjunction (**or**)

Negation (**not**)

If A is a statement, the statement "not A" (in symbols: $\neg A$) is **true** when A is **false**, and it is **false** when A is **true**.

Negation (**not**)

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Example

$\neg(3 > 4)$ is equivalent to $3 \leq 4$

"3 *is* not *greater than* 4" is equivalent to "3 *is less or equal than* 4"

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Conjunction (**and**)

The statement "A and B" (in symbols: $A \wedge B$) is **true** when both A and B are **true**, and it is **false** if at *at least* one of them is **false**.

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Example

"$(3 < 4) \wedge$ (5 is an odd number)" is **true**

Example

"(Today is Monday) $\wedge$ (we are in France)" is **false**

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Disjunction (**or**)

The statement "A or B" (in symbols: $A \vee B$) is **true** when at least one of A and B is **true**, and it is **false** if both of them are **false**.

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Disjunction (**or**)

The statement "A or B" (in symbols: $A \vee B$) is **true** when at least one of A and B is **true**, and it is **false** if both of them are **false**.

Example

"$(3 = 4) \vee$ (5 is an even number)" is **false**

Example

"(Today is Monday) $\vee$ (we are in Luxembourg)" is **true**

- **Important:** $\vee$ is always *inclusive*:

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Example (Another bad joke)

Waiter: "Would you like cheese or dessert?"
Mathematician: "Yes."

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Logical operations

- **Important:** $\vee$ is always *inclusive*:

Example (Another bad joke)

Waiter: "Would you like cheese or dessert?"
Mathematician: "Yes."

- $\neg$ has precedence over $\wedge$ and $\vee$:

$$\neg A \wedge B \text{ means } (\neg A) \wedge B, \qquad \neg A \vee B \text{ means } (\neg A) \vee B$$

(or just use parenthesis)

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Properties

If A, B and C are statements:

$A \wedge B = B \wedge A$	$A \vee B = B \vee A$	**commutativity**
$A \wedge (B \wedge C) = (A \wedge B) \wedge C$	$A \vee (B \vee C) = (A \vee B) \vee C$	**associativity**
$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$		**distributivity**
$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$		**distributivity***
$\neg(\neg A) = A$		**double negation**
$A \wedge \textbf{true} = A$	$A \wedge \textbf{false} = \textbf{false}$	
$A \vee \textbf{true} = \textbf{true}$	$A \vee \textbf{false} = A$	
$(\neg A) \wedge A = \textbf{false}$	$(\neg A) \vee A = \textbf{true}$	
$\neg(A \wedge B) = (\neg A) \vee (\neg B)$	$\neg(A \vee B) = (\neg A) \wedge (\neg B)$	**De Morgan's laws**

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Boolean algebra

- For simplicity: $\textbf{true} = 1$, $\textbf{false} = 0$

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- We have a set $\{0, 1\}$ with some operations $(\wedge, \vee, \neg)$

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- For simplicity: **true** $= 1$, **false** $= 0$
- We have a set $\{0, 1\}$ with some operations $(\wedge, \vee, \neg)$
- This is called a **Boolean algebra**

Truth tables

A compact way of describing an operator, or a composition of operators

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Example:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$(A \vee B) \wedge (\neg A)$
0	0	1	0	0	0
0	1	1	0	1	1
1	0	0	0	1	0
1	1	0	1	1	0

Truth tables

We can check that two statements are equivalent with truth tables

A	B	$\neg(A \wedge B)$	$(\neg A) \vee (\neg B)$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

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Implication

- "$A \implies B$" means *"If A (is true), then B (is true)"*

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Example

"If it rains, I will bring an umbrella"

(It rains) $\implies$ (I will bring an umbrella)

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"If it rains, I will bring an umbrella"
(It rains) $\implies$ (I will bring an umbrella)

Example

"If my grandpa had wheels, he would be a bike"
(My grandpa has wheels) $\implies$ (My grandpa is a bike)

Implication

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A	B	$A \implies B$	
0	0	1	No rain, I don't bring an umbrella
0	1	1	No rain, I bring an umbrella anyway
1	0	0	It rains, I don't bring an umbrella
1	1	1	It rains, I bring an umbrella

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Remark

"**false** $\implies A$" is always **true**, whatever A is (*ex falso quodlibet*)
"$A \implies$ **true**" is always true, whatever A is

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Notation

Sometimes we use the following symbols:

- “$A \impliedby B$” is the same as “$B \implies A$”
- “$A \iff B$ is the same as “$(A \implies B) \wedge (B \implies A)$”.
  It is read “A is equivalent to B” or “A if and only if B”.

Contrapositive

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Contrapositive

- The statement $(\neg B) \implies (\neg A)$ is called *contrapositive* of $A \implies B$
- It is equivalent to "$A \implies B$"
- Two proofs:
  1. Properties of logical operations
  2. Truth tables

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End of part 1

See you tomorrow!

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Quantifiers

Let S be a set and let $A(x)$ be a "variable statement" that depends on $x \in S$ (for example $S = \mathbb{N}$ and $A(x) =$"x is an even number").

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- **Universal quantifier** ($\forall$ or "for all"): "$\forall x \in S$, $A(x)$" means that if we replace x with any element of S, $A(x)$ is always **true**.
- **Existential quantifier** ($\exists$ or "there exists"): "$\exists x \in S$, $A(x)$" means that $A(x)$ is **true** for at least one value of x is S.

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Quantifiers - examples

Example

$S =$ "the set of all cars", $A(x)=$ "x is red"
$\forall x \in S,\ A(X)$ is **false**.
$\exists x \in S,\ A(X)$ is **true**.

Example

$S = \mathbb{N},\ A(x) = x > 5$
$\forall x \in S,\ A(x)$ is **false**.
$\exists x \in S,\ A(X)$ is **true**.

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Negation of quantifiers

Today's most important fact:

$$\neg(\forall x \in S,\, A(x)) = \quad ?$$
$$\neg(\exists x \in S,\, A(x)) = \quad ?$$

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Negation of quantifiers

Today's most important fact:

$$\neg(\forall x \in S,\ A(x)) = \exists x \in S,\ \neg A(x)$$
$$\neg(\exists x \in S,\ A(x)) = \quad ?$$

Example

$\neg$"every number is even" $=$ "there is at least one odd number"

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Today's most important fact:

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Example

$\neg$ "every number is even" $=$ "there is at least one odd number"

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(exercise)

Proofs

- A proof is a sequence of statements, each one logically deriving from the previous.

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- A proof is a sequence of statements, each one logically deriving from the previous.
- Proofs are used to derive new statements from statements that are known to be true.
- If A is known to be true and the implication $A \implies B$ is logically clear, then also B must be true.
- Every mathematical theorem must be justified with a proof.

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Example: direct proof

Theorem

The sum of two even numbers is even.

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Proof.

1. Recall the definition: a natural number x is called *even* if there is some natural number n such that $x = 2n$.

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Example: direct proof

Theorem

The sum of two even numbers is even.

Proof.

1. Recall the definition: a natural number x is called *even* if there is some natural number n such that $x = 2n$.
2. If x and y are even numbers, then there are natural numbers n and m such that $x = 2n$ and $y = 2m$.

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Proof.

1. Recall the definition: a natural number x is called *even* if there is some natural number n such that $x = 2n$.
2. If x and y are even numbers, then there are natural numbers n and m such that $x = 2n$ and $y = 2m$.
3. Then $x + y = 2n + 2m = 2(n + m)$.

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3. Then $x + y = 2n + 2m = 2(n + m)$.
4. Then $x + y$ is even.

$\square$

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Proofs by contradiction

Idea: I want to show $A =$ **true**. I show that the implication "$(\neg A) \implies$ **false**" is **true**. Then $\neg A =$ **false**, so $A =$ **true**.

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Definition

A natural number is called *prime* if it is different from 1 and it is only divisible by 1 and itself.

Theorem

There are infinitely many prime numbers.

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Proofs by contradiction

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1. Assume that there are only finitely many prime numbers.

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1. Assume that there are only finitely many prime numbers.
2. So there are n prime numbers, for some number n. Call them $p_1, p_2, \ldots, p_n$.

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1. Assume that there are only finitely many prime numbers.
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3. Let $u = p_1 \times p_2 \times \cdots \times p_n + 1$.

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3. Let $u = p_1 \times p_2 \times \cdots \times p_n + 1$.
4. u is not divisible by any of the prime numbers $p_1, \dots, p_n$.

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3. Let $u = p_1 \times p_2 \times \cdots \times p_n + 1$.
4. u is not divisible by any of the prime numbers $p_1, \ldots, p_n$.
5. Therefore u is only divisible by 1 and itself. So u is prime.

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Proofs by contradiction

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1. Assume that there are only finitely many prime numbers.
2. So there are n prime numbers, for some number n. Call them $p_1, p_2, \dots, p_n$.
3. Let $u = p_1 \times p_2 \times \cdots \times p_n + 1$.
4. u is not divisible by any of the prime numbers $p_1, \dots, p_n$.
5. Therefore u is only divisible by 1 and itself. So u is prime.
6. So $p_1, \dots, p_n$ are not the only prime numbers.

$\square$

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If I want to prove $\forall n \in \mathbb{N}, A(n)$:

1. Prove $A(0)$ (*base step*)
2. Prove $\forall n \in \mathbb{N}, (A(n) \implies A(n+1))$ (*inductive step*)

Proofs by induction

If I want to prove $\forall n \in \mathbb{N}, A(n)$:

1. Prove $A(0)$ (*base step*)
2. Prove $\forall n \in \mathbb{N}, (A(n) \implies A(n+1))$ (*inductive step*)

Theorem (Sum of natural numbers)

$$\forall n \in \mathbb{N}, \quad 0 + 1 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

1. Base case: $0 = 0$.

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2. Let n be any natural number.

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Proof.

1. Base case: $0 = 0$.
2. Let n be any natural number.
   If $A(n) =$ **false**, then $A(n) \implies A(n+1)$ is **true**.

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Proof.

1. Base case: $0 = 0$.
2. Let n be any natural number.
If $A(n) = \textbf{false}$, then $A(n) \implies A(n+1)$ is **true**.
If $A(n) = \textbf{true}$, we have to show that $A(n+1) = \textbf{true}$.

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Proof.

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   If $A(n) = \textbf{true}$, we have to show that $A(n+1) = \textbf{true}$.

$$
\begin{aligned}
0 + \cdots + (n+1) &= (0 + \cdots + n) + (n+1) = \\
&= \tfrac{n(n+1)}{2} + (n+1) = \qquad \text{since } A(n) = \textbf{true} \\
&= \tfrac{n^2+n+2n+2}{2} \\
&= \tfrac{(n+1)(n+2)}{2}
\end{aligned}
$$

so $A(n+1) = \textbf{true}$

$\square$

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