Elementary Logic exercises (Prep Camp 2020)

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September 6, 2021

1 Logical operations

Exercise 1.1. Determine if the following statements are **true** or **false**:

1. "Today is Tuesday or Germany has more inhabitants than Luxembourg"
2. "7 is odd and $2 + 2 = 5$"
3. Every number of the form $2^{2^n} + 1$, for $n = 1, 2, 3...$, is prime.

Solution.

1. **True**: regardless of when you solve this exercise, Germany has more inhabitants than Luxembourg.
2. **True**
3. **False**: the number $2^{2^5} + 1 = 4294967297 = 641 \times 6700417$ is not prime.

□

Exercise 1.2. What is the negation of the sentence "*I payed attention in class and I did not do my homework*" ?

Solution. "*I did not pay attention in class* ***or*** *I did my homework*" □

Exercise 1.3. Simplify the following logical expressions using the properties of logical operations (where A, B and C are statements):

1. $A \wedge (A \vee B)$
2. $A \vee (B \wedge A)$
3. $(A \vee B) \wedge \neg A$
4. $A \vee (\neg A \wedge B)$
5. $(\neg(A \vee \neg B)) \wedge ((A \vee C) \wedge \neg C)$

Solution. They are equivalent to the following (you can check with truth tables):

1. A
2. A
3. $B \wedge \neg A$
4. $A \vee B$
5. Let's do this one in more steps:

$$\begin{aligned}(\neg(A \vee \neg B)) \wedge ((A \vee C) \wedge \neg C) =& (\neg A \wedge B) \wedge ((A \vee C) \wedge \neg C) = \\ =& (\neg A \wedge B) \wedge ((A \wedge \neg C) \vee (C \wedge \neg C)) = \\ =& (\neg A \wedge B) \wedge ((A \wedge \neg C) \vee \textbf{false}) = \\ =& (\neg A \wedge B) \wedge (A \wedge \neg C) = \\ =& A \wedge \neg A \wedge B \wedge \neg C = \\ =& \textbf{false}\end{aligned}$$

□

2 Implication

Exercise 2.1. Fill in the following truth table:

A	B	C	$\neg(A \implies B)$	$(A \implies B) \implies C$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	0	1

Exercise 2.2 (Transitivity)**.** Prove that the following statement is true for any statements A, B and C:

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$$

Solution. Let's rewrite the first part in terms of basic logical operations:

$$(A \implies B) \wedge (B \implies C) = (B \vee \neg A) \wedge (C \vee \neg B)$$

now we can write a truth table for the two parts

A	B	C	$(B \vee \neg A) \wedge (C \vee \neg B)$	$A \implies C$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	0	0
1	0	1	0	1
1	1	0	0	0
1	1	1	1	1

With the help of the truth table we see that whenever the first part "$(\neg(A \vee \neg B)) \wedge ((A \vee C) \wedge \neg C)$" is true, also the implication "$A \implies C$" is true. This shows that the "big implication" is true. (If you are not convinced, you can add more details to this proof, for example by writing more truth tables.) ☐

Exercise 2.3. What is the contrapositive of "*If this table is not reserved, we sit here*" ?

Solution. "*If we do not sit here, this table is reserved*". One could also say this in another way, for example "*We do not sit here because this table is reserved*". ☐

3 Quantifiers

Exercise 3.1. Write the negation of the following statements:

1. $\exists x \in \mathbb{N},\ x^2 - 2 = 0$
2. "Every prime number is odd"
3. "Every person I have met likes pizza"
4. "There is at least one number greater than 7"
5. $\forall x \in \mathbb{N},\ x \geq 0$
6. $\forall x \in \mathbb{Z},\ (\exists y \in \mathbb{Z},\ x + y = 0)$

Solution. 1. $\forall x \in \mathbb{N},\, x^2 - 2 \neq 0$

2. "There is at least one prime number which is even"

3. "I have met at least one person that does not like pizza"

4. "Every number is less or equal than 7"

5. $\exists x \in \mathbb{N},\, x < 0$

6. $\exists x \in \mathbb{Z},\, (\forall y \in \mathbb{Z},\, x + y \neq 0)$

□

Exercise 3.2. There is another quantifier that we did not cover in the lecture, namely $\exists!$ (read "there exists exactly one"). For example, the sentence "*there exists exactly one natural number x such that x+2=5*" can be written in symbols as "$\exists! x \in \mathbb{N},\, x + 2 = 5$".

In this exercise, your task is to give a formal definition of this quantifier using the logical symbols that we have defined in class. In particular, you will need the following:

- the universal ($\forall$) and existential ($\exists$) quantifiers
- the conjunction $\wedge$
- the implication $\implies$

Moreover, you will need the equality symbol $=$ between two elements of a set (if a and b are two elements of the same set, "$a = b$" is a mathematical statement and it is **true** if and only if a and b are the same element).

Warning: your definition must depend on a set S and on a "variable statement" $A(x)$, as the existential and universal quantifiers.

Solution. The idea is that we want to write "*there is $x \in S$ such that $A(x)$ is true,* ***and****, for any other $y \in S$, $A(y)$ is false.* From this we see that the structure of the statement is

$$\exists x \in S,\, (\text{“something”} \wedge \text{“something else”})$$

The "something" part is just $A(x)$. The "something else" part can be written in different ways, for example

$$\forall y \in S,\, (y \neq x \implies \neg A(y)) \quad \text{or} \quad \forall y \in S,\, (A(y) \implies y = x)$$

(notice that the two implications above are one the contrapositive of the other); or also

$$\forall y \in S \setminus \{x\},\, \neg A(y)$$

So in conclusion, one way to define "$\exists!$" is the following:

$$\exists! x \in S,\, A(x) \quad := \quad \exists x \in S,\, (A(x) \wedge (\forall y \in S,\, (A(y) \implies y = x)))$$

□

4 Proofs

Exercise 4.1. Prove by induction that

$$\forall n \in \mathbb{N}, \quad \sum_{k=1}^{n} (2k - 1) = n^2$$

(here $\sum_{k=1}^{n}(2k-1)$ means $1 + 3 + 5 + \cdots + (2n - 1)$).

Solution. **Base case:** for $n = 0$ the sum is empty, so we have $0 = 0$ which is true. (If you do not think that the formula makes sense for $n = 0$, you can do prove it for $n \geq 1$ and fo $n = 1$ as a base case.)

Inductive step: we can assume that the formula works for a generic (but fixed) $n \in \mathbb{N}$ and prove that then it also works for $n + 1$. So:

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1) &= \left(\sum_{k=1}^{n}(2k-1)\right) + 2(n+1) - 1 = \\
&= n^2 + 2(n+1) - 1 = \\
&= n^2 + 2n + 1 = \\
&= (n+1)^2
\end{aligned}$$

which is the formula for $n + 1$. □

Exercise 4.2. If $n \in \mathbb{N}$ the *factorial* of n, denoted by $n!$ is defined as follows:

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n \times (n-1)! & \text{if } n > 0. \end{cases}$$

Prove by induction that if $n \geq 4$ then $n! \geq 2^n$.

Solution. **Base case:** here the base case is $n = 4$, and we have $4! = 24 \geq 16 = 2^4$.

Inductive step: we can assume that the inequality is true for n, and prove that it is also true for $n + 1$. We have:

$$(n+1)! = (n+1) \times n! \geq (n+1) \times 2^n \geq 2^{n+1}$$

□

Exercise 4.3. Is the following statement true or false? Give a proof of your answer.

$$\forall n \in \mathbb{N},\ n^2 - 4n + 5 > n$$

Solution. The statement is **false**. Since it starts with a universal quantifier, in order to prove that it is false we just need to provide one example of $n \in \mathbb{N}$ which makes it false. In other words, we need to prove that

$$\exists n \in \mathbb{N},\ n^2 - 4n + 5 \leq n$$

and a proof of this fact is very simple: for $n = 2$ we have $2^2 - 4 \times 2 + 5 = 1 \leq 2$. □

Exercise 4.4. Do the last point of Exercise 1.1 again, but this time give a proof of your answer.

Solution. Again, since we have to prove that the statement is false, we just need to show one counterexample. for example, the number $2^{2^5} + 1 = 4294967297 = 641 \times 6700417$ is not prime. □