Elementary Logic exercises (Prep Camp 2020)

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1 Logical operations

Exercise 1.1. Determine if the following statements are **true** or **false**:

1. "Today is Tuesday or Germany has more inhabitants than Luxembourg"
2. "7 is odd and $2 + 2 = 5$"
3. Every number of the form $2^{2^n} + 1$, for $n = 1, 2, 3...$, is prime.

Exercise 1.2. What is the negation of the sentence "*I payed attention in class and I did not do my homework*" ?

Exercise 1.3. Simplify the following logical expressions using the properties of logical operations (where A, B and C are statements):

1. $A \wedge (A \vee B)$
2. $A \vee (B \wedge A)$
3. $(A \vee B) \wedge \neg A$
4. $A \vee (\neg A \wedge B)$
5. $(\neg(A \vee \neg B)) \wedge ((A \vee C) \wedge \neg C)$

2 Implication

Exercise 2.1. Fill in the following truth table:

A	B	C	$\neg(A \implies B)$	$(A \implies B) \implies C$
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Exercise 2.2 (Transitivity)**.** Prove that the following statement is true for any statements A, B and C:

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C)$$

Exercise 2.3. What is the contrapositive of "*If this table is not reserved, we sit here*" ?

3 Quantifiers

Exercise 3.1. Write the negation of the following statements:

1. $\exists x \in \mathbb{N},\ x^2 - 2 = 0$
2. “Every prime number is odd”
3. “Every person I have met likes pizza”
4. “There is at least one number greater than 7”
5. $\forall x \in \mathbb{N},\ x \geq 0$
6. $\forall x \in \mathbb{Z},\ (\exists y \in \mathbb{Z},\ x + y = 0)$

Exercise 3.2. There is another quantifier that we did not cover in the lecture, namely $\exists!$ (read “there exists exactly one”). For example, the sentence “*there exists exactly one natural number x such that x+2=5*” can be written in symbols as “$\exists! x \in \mathbb{N},\ x + 2 = 5$”.

In this exercise, your task is to give a formal definition of this quantifier using the logical symbols that we have defined in class. In particular, you will need the following:

- the universal ($\forall$) and existential ($\exists$) quantifiers
- the conjunction $\wedge$
- the implication $\implies$

Moreover, you will need the equality symbol $=$ between two elements of a set (if a and b are two elements of the same set, “$a = b$” is a mathematical statement and it is **true** if and only if a and b are the same element).

Warning: your definition must depend on a set S and on a “variable statement” $A(x)$, as the existential and universal quantifiers.

4 Proofs

Exercise 4.1. Prove by induction that

$$\forall n \in \mathbb{N}, \quad \sum_{k=1}^{n} (2k - 1) = n^2$$

(here $\sum_{k=1}^{n}(2k-1)$ means $1 + 3 + 5 + \cdots + (2n - 1)$).

Exercise 4.2. If $n \in \mathbb{N}$ the *factorial* of n, denoted by $n!$ is defined as follows:

$$n! = \begin{cases} 1 & \text{if } n = 0, \\ n \times (n-1)! & \text{if } n > 0. \end{cases}$$

Prove by induction that if $n \geq 4$ then $n! \geq 2^n$.

Exercise 4.3. Is the following statement true or false? Give a proof of your answer.

$$\forall n \in \mathbb{N},\ n^2 - 4n + 5 > n$$

Exercise 4.4. Do the last point of Exercise 1.1 again, but this time give a proof of your answer.