X2-StudentsRequests-notebook

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1 Diffie-Hellman key exchange

The following is a simple implementation of the classic Diffie-Hellman key exchange cryptographic protocol.

[1]:
```
# Public information:
p = Primes()[10^3 + randint(1,10000)] # random prime
g = randint(2, p-1) # random integer

print("Public key: p =", p, "and g =", g, "\n")

a = randint(2, p-1) # Only Alice knows this
b = randint(2, p-1) # Only Bob knows this

print("[[ Alice's secret key: a =", a, "]]")
print("[[ Bob's secret key: b =", b, "]]", "\n")

h1 = (g^a) % p # Alice sends this to Bob
h2 = (g^b) % p # Bob sends this to Alice

print("Alice sends h1 =", h1, "to Bob")
print("Bob sends h2 =", h2, "to Alice", "\n")

secret_a = (h2^a) % p # Alice can compute this because she knows a
secret_b = (h1^b) % p # Bob can compute this because he knows b

print("Alice computed", secret_a, "using h2 and her secret a")
print("Bob computed", secret_b, "using h1 and his secret b")
```

```
Public key: p = 75521 and g = 58258

[[ Alice's secret key: a = 22794 ]]
[[ Bob's secret key: b = 69773 ]]

Alice sends h1 = 31067 to Bob
Bob sends h2 = 54398 to Alice

Alice computed 30031 using h2 and her secret a
```

```
Bob computed 30031 using h1 and his secret b
```

1.1 General Diffie-Hellman

The following code is an implementation of a generic Diffie-Hellman key exchange protocol that uses a group G instead of $(\mathbb{Z}/p\mathbb{Z})^\times$.

[2]:
```
def genericDH(G):
    if G.cardinality() == 1:
        print("Group is trivial, can't do anything")
        return
    g = G.random_element()
    while g == G.identity(): # Make sure g is not trivial
        g = G.random_element()

    print("Public key:\nG =", G, "\ng =", g, "\n")

    a = randint(2, G.exponent()-1) # Only Alice knows this
    b = randint(2, G.exponent()-1) # Only Bob knows this

    print("[[ Alice's secret key: a =", a, "]]")
    print("[[ Bob's secret key: b =", b, "]]", "\n")

    # "Ternary operator", I did not explain this
    # https://docs.python.org/3/reference/expressions.
↪html#conditional-expressions
    h1 = g^a if G.is_multiplicative() else a*g # Alice sends this to Bob
    h2 = g^b if G.is_multiplicative() else b*g # Bob sends this to Alice

    print("Alice sends h1 =", h1, "to Bob")
    print("Bob sends h2 =", h2, "to Alice", "\n")

    secret_a = h2^a if G.is_multiplicative() else a*h2 # Alice can compute this␣
↪because she knows a
    secret_b = h1^b if G.is_multiplicative() else b*h1 # Bob can compute this␣
↪because he knows b

    print("Alice computed", secret_a, "using h2 and her secret a")
    print("Bob computed", secret_b, "using h1 and his secret b")

E = EllipticCurve(GF(157), [1,-1])
G = E.abelian_group()
genericDH(G)
```

```
Public key:
G = Additive abelian group isomorphic to Z/171 embedded in Abelian group of
points on Elliptic Curve defined by y^2 = x^3 + x + 156 over Finite Field of
size 157
```

```
g = (53 : 90 : 1)

[[ Alice's secret key: a = 145 ]]
[[ Bob's secret key: b = 65 ]]

Alice sends h1 = (150 : 80 : 1) to Bob
Bob sends h2 = (4 : 58 : 1) to Alice

Alice computed (28 : 28 : 1) using h2 and her secret a
Bob computed (28 : 28 : 1) using h1 and his secret b
```

2 Numerical methods for differential equations

2.1 Euler's method (ODE)

In sage you can use `ode_solver()` to solve any ordinary differential equation by hand, but Euler's method is very simple to implement by hand:

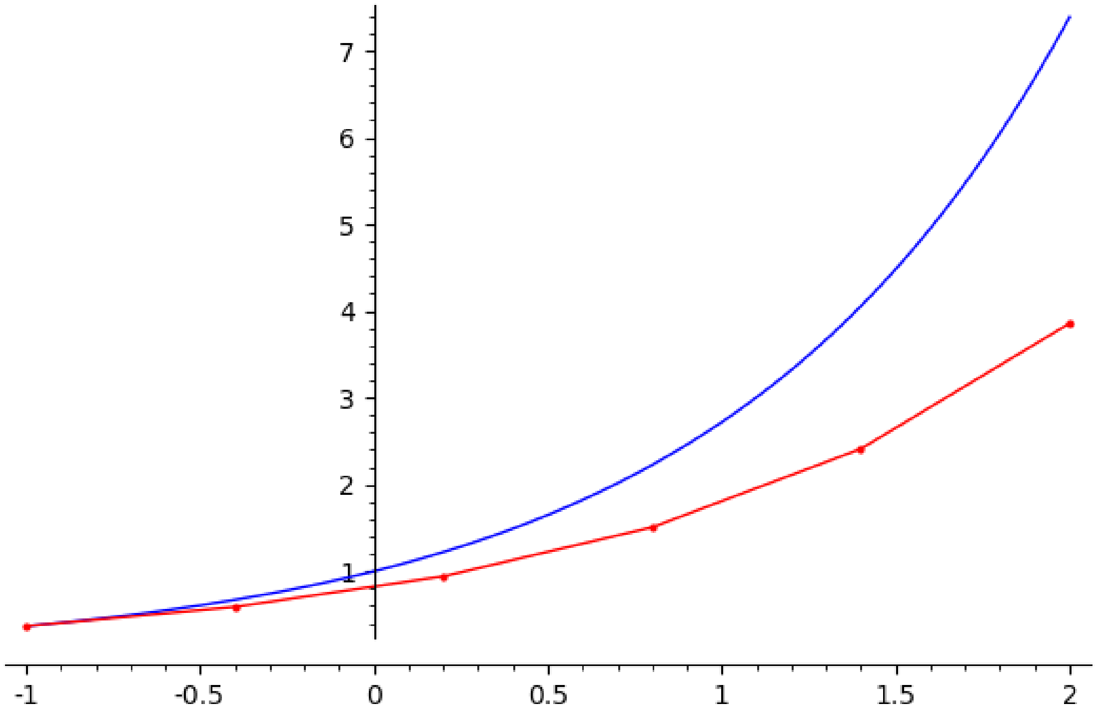
[4]:
```
var('y')

def euler_desolve(f, x0, y0, x1):
    n = 5
    h = (x1-x0)/n
    S = []
    Y = [y0]
    for i in range(n+1):
        S.append(x0 + i*h)
        Y.append(N(  Y[i] + h*f(S[i], Y[i])  ))
    return S, Y

f(x,y) = y
x0 = -1
x1 = 2
y0 = e^(-1)

S, Y = euler_desolve(f, x0, y0, x1)
plot(e^x, -1, 2) + line([(S[i], Y[i]) for i in range(len(S))], color='red',␣
 ↪marker='o', markersize=2)
```

[4]:



Sage also has an eulers_method() function ``for pedagogical purposes only'':

[5]:
```
# Usage: eulers_method(f, x0, y0, h, x1)
eulers_method(f, -1, N(e^(-1)), 0.1, 2)
```

```
         x                    y                  h*f(x,y)
        -1     0.367879441171442   0.0367879441171442
-0.900000000000000     0.404667385288587   0.0404667385288587
-0.800000000000000     0.445134123817445   0.0445134123817445
-0.700000000000000     0.489647536199190   0.0489647536199190
-0.600000000000000     0.538612289819109   0.0538612289819109
-0.500000000000000     0.592473518801020   0.0592473518801020
-0.400000000000000     0.651720870681122   0.0651720870681122
-0.300000000000000     0.716892957749234   0.0716892957749234
-0.200000000000000     0.788582253524157   0.0788582253524157
-0.100000000000000     0.867440478876573   0.0867440478876573
-1.38777878078145e-16     0.954184526764230   0.0954184526764230
0.0999999999999999      1.04960297944065     0.104960297944065
0.200000000000000      1.15456327738472     0.115456327738472
0.300000000000000      1.27001960512319     0.127001960512319
0.400000000000000      1.39702156563551     0.139702156563551
0.500000000000000      1.53672372219906     0.153672372219906
0.600000000000000      1.69039609441897     0.169039609441897
0.700000000000000      1.85943570386086     0.185943570386086
```

0.800000000000000	2.04537927424695	0.204537927424695
0.900000000000000	2.24991720167165	0.224991720167165
1.00000000000000	2.47490892183881	0.247490892183881
1.10000000000000	2.72239981402269	0.272239981402269
1.20000000000000	2.99463979542496	0.299463979542496
1.30000000000000	3.29410377496746	0.329410377496746
1.40000000000000	3.62351415246420	0.362351415246420
1.50000000000000	3.98586556771062	0.398586556771062
1.60000000000000	4.38445212448168	0.438445212448168
1.70000000000000	4.82289733692985	0.482289733692985
1.80000000000000	5.30518707062284	0.530518707062284
1.90000000000000	5.83570577768512	0.583570577768512
2.00000000000000	6.41927635545363	0.641927635545363

2.2 Solving the heat equation with a finite difference method

[ ]:
```
def heat_fdm(u0j, u1j, ui0):
    m, n = len(u0j)-1, len(ui0)-1
    k, h = 1/m, 1/n

    u = [[0] * (m+1) for i in range(n+1)]
    for j in range(m+1):
        u[0][j] = u0j[j]
    for j in range(m+1):
        u[n][j] = u1j[j]
    for i in range(n+1):
        u[i][0] = ui0[i]

    for j in range(0,m):
        for i in range(1,n):
            u[i][j+1] = (k/(h*h)) * (u[i+1][j] - 2*u[i][j] + u[i-1][j]) +␣
 ↪u[i][j]

    return u

n, m = 20, 20
u0j = [10 - (j/m)*10 for j in range(m+1)] # One extreme goes from hot to cold
u1j = [(j/m)*10 for j in range(m+1)] # The other does the opposite
ui0 = [10 - (i/m)*10 for i in range(0,n+1)]

u = heat_fdm(u0j, u1j, ui0)
for t in range(m+1):
    show(line([(i/n, u[i][t]) for i in range(n+1)], ymin=-1, ymax =12))
```