Mathematical Software - Homework 4

Deadline: Sunday, June 6

For this exercise you should have received this text in .ipynb format. Complete the exercises by modifying this file, and submit the modified version

Exercise 1 (6 points)

Use Sage to find the intersection points *in the real plane* (that is, only those points such that *both* coordinates are real numbers) of the following pairs of geometric objects:

- The circle of equation $x^2 + y^2 = 4$ and the ellipse of equation $\left(\frac{x}{2}\right)^2 + (2y)^2 = 4$.
- The circle of equation $x^2 + y^2 = 4$ and the ellipse of equation $\left(\frac{x}{2} - 2\right)^2 + (2y)^2 = 4$.
- The curve of equation $y^2 = x^3 - x + 1$ and the horizontal line $y = 10$.
- The x-axis and the graph of the function $f(x) = \log(x) - e^{-x}$. *Hint: $f(x)$ has only one real zero.*

Exercise 2 (6 points)

(a) Use Sage to compute

- the derivative
- a primite (i.e. integral)
- the power series expansion around 0 up to order 4

of the following functions:

- $f(x) = e^x$
- $f(x) = \sin(x)$
- $f(x) = \cos(x)$
- $f(x) = \tan(x)$
- $f(x) = \log(1 + x)$
- $f(x) = \sqrt[3]{1 + x}$

(b) Use Sage to get the Latex code that represents the objects you computed above.

(c) Arrange the results of the previous points in a table in Latex. The table should have 4 columns (function, derivative, integral, series) and one row for each of the functions above. *Note: when including Latex in a Markdown cell in Jupyter you will not receive any warning if you make mistakes; instead the Latex will simply not be rendered and it will appear as plain text. If you have troubles making this work you can send me a separate .tex (and .pdf) file.*

Exercise 3 (4 points)

The equation

$$y^2 + x^{16} = 1$$

determines a closed curve in $\mathbb{R}^2$ that looks like a rounded square. Determine the area of that shape, giving both an exact value (which might depend on some functions that Sage knows, but you don't) and an approximate value.

Exercise 4 (12 points)

A team of biologists is monitoring the population of river shrimps in the Alzette. At first they thought that the size $P(t)$ of their population on day t would satisfy the differential equation $P'(t) = P(t)/10$. However this does not work well with the data they have collected, so they now believe that the population of shrimps follows the formula $P'(t) = P(t)/10 - b$ for some value of b between 1 and 100. They need your help here.

(a) Using Sage, find a solution for the differential equation with initial conditions

$$\begin{cases} P'(t) & = \frac{P(t)}{10} - b \\ P(1) & = 1000 \end{cases}$$

where b is a generic constant.

(b) The list `data` in the cell below contains the actual number of shrimps that was measured every day from day 1 (the 0 at the beginning is meaningless, but it will help to keep it there). Plot in one single picture, possibly using different colors for each:

- The data as a bar chart.
- A curve that interpolates the data, using one of the methods shown in class.
- The solution of the differential equation for $b = 0$.
- The solution of the differential equation for a value of b of your choice ($1 \le b \le 100$) that fits the data better than $b = 0$. *(For this last point there is no right or wrong choice, just pick one that looks good)*

[ ]:
```
data = [0, 1000, 1123, 1223, 1190, 1432, 1553, 1709, 1826, 1980, 2146, 2172,
↪2383, 2588, 2822, 3401, 3330, 4157, 3994, 4995, 5392, 5910, 6468, 7128, 7325,
↪7984, 9634, 10473, 11761, 12777]
```

Grading

This homework assignment is worth 28 ($24 + 4$) points, distributed as described above.

Your final grade for the course will be the total of points you obtained (notice that the maximum is $20 + 20 + 16 + 28 = 84$) divided by 4, rounded to the nearest integer. More precisely

$$\text{grade} = \min\left(20, \left\lfloor \frac{\text{total}}{4} + 0.5 \right\rfloor\right)$$