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THE GENERAL LINEAR GROUP OVER A RING

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ABSTRACT. Let m be any positive integer, R be a ring with identity, $M_m(R)$ be the matrix ring of all m by m matrices over R and $G_m(R)$ be the multiplicative group of all m by m nonsingular matrices in $M_m(R)$. In this paper, the following are investigated: (1) for any pairwise coprime ideals $\{I_1, I_2, \ldots, I_n\}$ in a ring R, $M_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n))$ is isomorphic to $M_m(R/I_1) \times M_m(R/I_2) \times \cdots \times M_m(R/I_n)$, and so $G_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n))$ is isomorphic to $G_m(R/I_1) \times G_m(R/I_2) \times \cdots \times G_m(R/I_n)$; (2) In particular, if R is a finite ring with identity, then the order of $G_m(R)$ can be computed.

1. Introduction

Throughout this paper all rings are assumed to be rings with identity. Let I be an ideal in a ring R and $a, b \in R$. Recall that a is said to be congruent to b modulo I (denoted $a \equiv b \pmod{I}$) if $a - b \in I$. Clearly, the congruence relation is an equivalence relation on R. Two ideals $I, I^{'}$ of R are *coprime* if $I + I^{'} = R$. A set of nonzero ideals $\{I_1, I_2, \ldots, I_n\}$ in a ring R is *pairwise coprime* if $I_j + I_k = R$ for all $j, k = 1, 2, \ldots, n$ $(j \neq k)$.

THEOREM 1.1. (Chinese Remainder Theorem) *Let $\{I_1, I_2, \ldots, I_n\}$ be pairwise coprime ideals in a ring R. If $b_1, b_2, \ldots, b_n \in R$, then there exists $b \in R$ such that $b \equiv b_i (\text{mod } I_i)$ $(i = 1, 2, \ldots, n)$. Furthermore, b is uniquely determined up to congruence modulo the ideal $I_1 \cap I_2 \cap \cdots \cap I_n$.*

Proof. See [1, Theorem 2.25]. □

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Corollary 1.2. *Let $\{I_1, I_2, \ldots, I_n\}$ be pairwise coprime ideals in a ring R. Then $R/(I_1 \cap I_2 \cap \cdots \cap I_n)$ is isomorphic to $R/I_1 \times R/I_2 \times \cdots \times R/I_n$ as rings.*

Proof. See [1, Corollary 2.27]. □

Remark 1. For any pairwise coprime ideals $\{I_1, I_2, \ldots, I_n\}$ in a commutative ring R, $I_1 \cap I_2 \cap \cdots \cap I_n = I_1 \cdot I_2 \cdots I_n$.

Let m be a positive integer and $M_m(R)$ be the matrix ring of all $m \times m$ matrices over a ring R. Consider the following relation $\equiv_m$ defined on $M_m(R)$: For any $A = [a_{ij}]$ and $B = [b_{ij}] \in M_m(R)$, $A \equiv_m B (\text{mod } I)$ (we read this A is congruent to B modulo I) if $a_{ij} \equiv b_{ij}$ (modulo I) for all $i, j = 1, 2, \ldots, m$ (i.e., a_{ij} - $b_{ij} \in I$). We can observe that the congruence relation $\equiv_m$ is an equivalence relation on $M_m(R)$ satisfying the following properties:
For any A, B, C and $D \in M_m(R)$ such that $A \equiv_m B (\text{mod } I)$ and $C \equiv_m D (\text{mod } I)$,

[1] $A + C \equiv_m B + D (\text{mod } I)$.

[2] $AC \equiv_m BD (\text{mod } I)$. In particular, $A^s \equiv_m B^s (\text{mod } I)$ for all positive integers s.

In this paper, we denote $G(R)$ by the multiplicative group of all units in R and $G_m(R)$ by the multiplicative group of all nonsingular matrices in $M_m(R)$.

Theorem 1.3. *Let m and n be any positive integers, R be a ring and $\{I_1, I_2, \ldots, I_n\}$ be pairwise coprime ideals in a ring R. If $A_1 = [a_{ij}^{(1)}], A_2 = [a_{ij}^{(2)}], \ldots, A_n = [a_{ij}^{(n)}] \in M_m(R)$, then there exists $A \in M_m(R)$ such that $A \equiv A_k (\text{mod } I_k)$ for all $k = 1, 2, \ldots, n$. Furthermore, A is uniquely determined up to congruence modulo the ideal $I_1 \cap I_2 \cap \cdots \cap I_n$.*

Proof. Since $a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(n)} \in R$ for all $i, j = 1, 2, \ldots, m$, there exists $a_{ij} \in R$ such that $a_{ij} \equiv a_{ij}^{(k)} (\text{mod } I_k)$ $(k = 1, 2, \ldots, n)$ by Theorem 1.1. Let $A = [a_{ij}] \in M_m(R)$. Then $A \equiv_m A_k$ $(\text{mod } I_k)(k = 1, 2, \ldots, n)$. Since a_{ij} is uniquely determined up to congruence modulo the ideal $I_1 \cap I_2 \cap \cdots \cap I_n$ for all $i, j = 1, 2, \ldots, m$, A is also uniquely determined up to congruence modulo the ideal $I_1 \cap I_2 \cap \cdots \cap I_n$. □

Corollary 1.4. *Let m and n be any positive integers, R be a ring and $\{I_1, I_2, \ldots, I_n\}$ be ideals in a ring R. Then there is a monomorphism* *of rings* $\theta : M_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n)) \longrightarrow M_m(R/I_1) \times M_m(R/I_2) \times \cdots \times M_m(R/I_n)$. *If* $\{I_1, I_2, \ldots, I_n\}$ *is pairwise coprime, then* θ *is an isomorphism.*

Proof. Consider a map $\theta_1 : M_m(R) \longrightarrow M_m(R/I_1) \times M_m(R/I_2) \times \cdots \times M_m(R/I_n)$ defined by $\theta_1([a_{ij}]) = ([a_{ij}+I_1], [a_{ij}+I_2], \ldots, [a_{ij}+I_n])$ for all $[a_{ij}] \in M_m(R)$. It is straightforward to show that θ_1 is a ring homomorphism and the kernel of θ_1 (denoted by $ker(\theta_1)$) is $M_m(I_1 \cap I_2 \cap \cdots \cap I_n)$. Since $M_m(R)/ker(\theta_1)$ is isomorphic to $M_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n))$, the map $\theta : M_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n)) \longrightarrow M_m(R/I_1) \times M_m(R/I_2) \times \cdots \times M_m(R/I_n)$ is a monomorphism. Suppose that $\{I_1, I_2, \ldots I_n\}$ is a pairwise coprime ideals in a ring R. To show that θ is an isomorphism, it is enough to show that θ is onto. Let $([a_{ij}^{(1)} + I_1], [a_{ij}^{(2)} + I_2], \ldots, [a_{ij}^{(n)} + I_n]) \in M_m(R/I_1) \times M_m(R/I_2) \times \cdots \times M_m(R/I_n)$ be arbitrary. Then by Theorem 1.3, there exists $[a_{ij}] \in M_m(R)$ such that $[a_{ij}] \equiv [a_{ij}^{(k)}]$(mod I_k) for all $k = 1, 2, \ldots, n$. Thus $\theta([a_{ij}] + I_1 \cap I_2 \cap \cdots \cap I_n) = ([a_{ij}^{(1)} + I_1], [a_{ij}^{(2)} + I_2], \ldots, [a_{ij}^{(n)} + I_n])$, and so θ is an isomorphism. $\square$

Corollary 1.5. *Let* m *and* k *be any positive integers,* $\mathbb{Z}_k$ *be the ring of integers modulo* k. *If* $p_1^{n_1} \cdot p_2^{n_2} \cdots p_s^{n_s}$ *is the prime factorization of* k, *then* $M_m(\mathbb{Z}_k)$ *is isomorphic to* $M_m(\mathbb{Z}_{p_1^{n_1}}) \times M_m(\mathbb{Z}_{p_2^{n_2}}) \times \cdots \times M_m(\mathbb{Z}_{p_s^{n_s}})$.

Proof. Let $I_i = p_i^{n_i}\mathbb{Z}$ be an ideal of $\mathbb{Z}$, the ring of integers, for all $i = 1, 2, \ldots, s$. Since $p_1^{n_1} \cdot p_2^{n_2} \cdots p_s^{n_s}$ is the prime factorization of k, the set of ideals $\{I_1, \ldots, I_s\}$ is pairwise coprime. Since $M_m(\mathbb{Z}/I_i)$ is isomorphic to $M_m(\mathbb{Z}_{p_i^{n_i}})$ for all $i = 1, 2, \ldots, s$, $M_m(\mathbb{Z}_k)$ is isomorphic to $M_m(\mathbb{Z}_{p_1^{n_1}}) \times M_m(\mathbb{Z}_{p_2^{n_2}}) \times \cdots \times M_m(\mathbb{Z}_{p_s^{n_s}})$ by Corollary 1.4. $\square$

Corollary 1.6. *Let* m *and* n *be any positive integers and* $\{I_1, I_2, \ldots, I_n\}$ *be ideals in a ring* R. *If* $\{I_1, I_2, \ldots, I_n\}$ *is pairwise coprime, then* $G_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n))$ *is isomorphic to* $G_m(R/I_1) \times G_m(R/I_2) \times \cdots \times G_m(R/I_n)$.

Proof. By Corollary 1.4, $M_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n))$ is isomorphic to $M_m(R/I_1) \times M_m(R/I_2) \times \cdots \times M_m(R/I_n)$. Since $G_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n))$, the multiplicative group of $M_m(R/I_1) \times M_m(R/I_2) \times \cdots \times M_m(R/I_n)$, is $G_m(R/I_1) \times G_m(R/I_2) \times \cdots \times G_m(R/I_n)$, $G_m(R/(I_1 \cap I_2 \cap \cdots \cap I_n))$ is isomorphic to $G_m(R/I_1) \times G_m(R/I_2) \times \cdots \times G_m(R/I_n)$. $\square$

Corollary 1.7. *Let m and k be any positive integers, $\mathbb{Z}_k$ be the ring of integers modulo k. If $p_1^{n_1} \cdot p_2^{n_2} \cdots p_s^{n_s}$ is the prime factorization of k, then $G_m(\mathbb{Z}_k)$ is isomorphic to $G_m(\mathbb{Z}_{p_1^{n_1}}) \times G_m(\mathbb{Z}_{p_2^{n_2}}) \times \cdots \times G_m(\mathbb{Z}_{p_s^{n_s}})$.*

Proof. It follows from Corollary 1.5 and Corollary 1.6. □

2. The order of $G_m(R)$ when R is a commutative ring

Let R be a finite commutative ring. In this section, we will compute the order of $G_m(R)$, the multiplicative group of all nonsingular matrices in $M_m(R)$ (called the general linear group of degree m over R) for all positive integers m. We will denote the order of $G_m(R)$ by $|G_m(R)|$. In [2], the following Theorem has been shown:

Theorem 2.1. *Let R be a finite commutative ring. Then R decomposes (up to order of summands) uniquely as a direct product of local rings. Precisely, $R \simeq (R/P_1^t) \times (R/P_2^t) \times \cdots \times (R/P_n^t)$ for some positive integers n and t, where $P_1, \ldots, P_n$ are all distinct prime (equally maximal) ideals of R.*

Proof. See [2, Theorem VI.2]. □

Lemma 2.2. *Let R and S be any two rings. Then $M_m(R \times S) \simeq M_m(R) \times M_m(S)$.*

Proof. Define $\phi : M_m(R \times S) \to M_m(R) \times M_m(S)$ by $\phi([(a_{ij}, b_{ij})]) = ([a_{ij}], [b_{ij}])$ for all $[(a_{ij}, b_{ij})] \in M_m(R \times S)$. Then it is straightforward to show that ϕ is an isomorphism. □

Corollary 2.3. *Let R be a finite commutative ring such that $R \simeq (R/P_1^t) \times (R/P_2^t) \times \cdots \times (R/P_n^t)$ for some positive integers n and t, where $P_1, P_2, \ldots, P_n$ are all distinct prime ideals of R given in Theorem 2.1. Then $G_m(R) \simeq G_m(R/P_1^t) \times G_m(R/P_2^t) \times \cdots \times G_m(R/P_n^t)$.*

Proof. It follows from Corollary 1.6 and Lemma 2.2. □

Corollary 2.4. *Let R be a finite commutative ring such that $R \simeq (R/P_1^t) \times (R/P_2^t) \times \cdots \times (R/P_n^t)$ for some positive integers n and t, where $P_1, P_2, \ldots, P_n$ are all distinct prime ideals of R given in Theorem 2.1. Then $|G_m(R)| = |G_m(R/P_1^t)| \cdot |G_m(R/P_2^t)| \cdots |G_m(R/P_n^t)|$.*

Proof. It follows from Corollary 2.3. □

To compute $|G_m(R)|$, by Corollary 2.4 it is enough to compute $|G_m(R/P_i^t)|$ for all $i = 1, \ldots, n$, where $P_1, P_2, \ldots, P_n$ are all distinct prime (equally maximal) ideals of R given in Theorem 2.1.

THEOREM 2.5. *Let R be a commutative ring and m be any positive integer. Then $A \in M_m(R)$ is invertible if and only if $|A|$, the determinant of $A \in R$, is a unit in R.*

Proof. See [1, Proposition 3.7]. □

LEMMA 2.6. *Let R be a commutative ring, P be an ideal of R and k ($k \geq 2$) be a positive integer. Then*

(1) *the map $\sigma : R/P^k \to R/P^{k-1}$ defined by $\sigma(a + P^k) = a + P^{k-1}$ for all $a + P^k \in R/P^k$ is a natural ring homomorphism.*

(2) *$\sigma|_{G(R/P^k)}$, the restriction of σ to $G(R/P^k)$, is a group homomorphism from $G(R/P^k)$ into $G(R/P^{k-1})$.*

(3) *In addition, if R is a local ring with the maximal ideal P, then $\sigma|_{G(R/P^k)}$ is onto.*

Proof. (1) Since $P^k \subseteq P^{k-1}$, the map $\sigma : R/P^k \to R/P^{k-1}$ defined by $\sigma(a + P^k) = a + P^{k-1}$ for all $a + P^k \in G(R/P^k)$ is well-defined. Clearly, σ is a ring homomorphism.

(2) For all $\bar{a} = a+P^k \in G(R/P^k)$, there exists $\bar{b} = b+P^k \in G(R/P^k)$ such that $\bar{a}\bar{b} = \bar{b}\bar{a} = \bar{1} = 1 + P^k$. Thus $1 - ab, 1 - ba \in P^k$. Since $P^k \subseteq P^{k-1}$, $1 - ab, 1 - ba \in P^{k-1}$, and then $a + P^{k-1} \in G(R/P^{k-1})$. Thus the map $\sigma|_{G(R/P^k)}$ is well-defined. For all $a + P^k, c + P^k \in G(R/P^k)$, $\sigma|_{G(R/P^k)}((a + P^k)(c + P^k)) = \sigma|_{G(R/P^k)}(ac + P^k) = ac + P^{k-1} = (a + P^{k-1})(c + P^{k-1})$. Hence $\sigma|_{G(R/P^k)}$ is a group homomorphism.

(3) Let $a + P^{k-1} \in G(R/P^{k-1})$ be arbitrary. Then there exists $b + P^{k-1} \in G(R/P^{k-1})$ such that $ab + P^{k-1} = (a + P^{k-1})(b + P^{k-1}) = (b+P^{k-1})(a+P^{k-1}) = ba+P^{k-1} = 1+P^{k-1}$. Thus $1-ab, 1-ba \in P^{k-1}$. Since $(R/P^k)/(P^{k-1}/P^k) \simeq R/P^{k-1}$ by the Third Isomorphism Theorem of Rings, without loss of generality we can let $(R/P^k)/(P^{k-1}/P^k) = R/P^{k-1}$, i.e., $(a + P^k) + (P^{k-1}/P^k) = \sigma(a + P^{k-1}) = a + P^{k-1}$ for all $a + P^{k-1} \in R/P^{k-1}$, where σ is a natural ring homomorphism given in (1). Since $ab + P^{k-1} = ba + P^{k-1} = 1 + P^{k-1}$, $ab - 1 + P^k, ba - 1 + P^k \in P^{k-1}/P^k$, and so $ab - 1, ba - 1 \in P^{k-1} \subseteq P$. Thus $ab, ba \in 1 + P$. Since R is a local ring with the maximal ideal P, $1 + P \subseteq G(R)$. Therefore, $a \in G(R)$, and so $a + P^k \in G(R/P^k)$. Therefore, $\sigma|_{G(R/P^k)}$ is onto. □

Theorem 2.7. *Let R be a finite local commutative ring, P be the unique maximal ideal of R and k be a positive integer. Then*

(1) *there exists a normal subgroup N of $G_m(R/P^k)$ such that $G_m(R/P^k)/N \simeq G_m(R/P^{k-1})$.*

(2) *$|G_m(R/P^k)| = (|P^{k-1}|/|P^k|)^{m^2} \cdot |G_m(R/P^{k-1})|$ for all positive integer m.*

(3) *$|G_m(R/P^k)| = (|P/P^k|)^{m^2} \cdot |G_m(R/P)|$ for all positive integer m, where $|G_m(R/P)| = (|R/P|^m - 1)(|R/P|^m - |R/P|) \cdots (|R/P|^m - |R/P|^{m-1})$.*

Proof. (1) Consider the map $\theta : G_m(R/P^k) \to G_m(R/P^{k-1})$ defined by $\theta([a_{ij} + P^k]) = [\sigma(a_{ij} + P^k)] = [a_{ij} + P^{k-1}]$ for all $[a_{ij} + P^k] \in G_m(R/P^k)$, where $\sigma|_{G(R/P^k)}$ is a group homomorphism given in Lemma 2.6. The map θ is well-defined. Indeed, for all $A = [a_{ij} + P^k] \in G_m(R/P^k)$, $|A| \in G(R/P^k)$ by Theorem 2.5, and also $|A| \in G(R/P^{k-1})$. It is easy to show that θ is a group homorphism. Next, we will show that θ is onto. Let $B = (b_{ij} + p^{k-1}) \in G_m(R/P^{k-1})$ be arbitrary. By Theorem 2.5, $|B| \in G(R/P^{k-1})$, where $|B|$ is the determinant of B. By Lemma 2.6, there exists $b_{ij} + P^k \in R/P^k$ such that $\sigma(b_{ij} + P^k) = b_{ij} + P^{k-1}$ for all $i, j = 1, \ldots, m$. Let $B_0 = [b_{ij} + P^k] \in M_m(R/P^k)$. Since $\sigma(|B_0|) = |B|$ and $|B| \in G(R/P^{k-1})$, $|B_0| \in G(R/P^k)$ and so $B_0 \in G_m(R/P^{k-1})$. Thus $\theta(B_0) = B$ and so θ is onto. Let $N = Ker(\theta)$. By the First Isomorphism Theorem of Groups, $G_m(R/P^k)/N \simeq G_m(R/P^{k-1})$.

(2) We can note that $ker(\theta) = \{[a_{ij} + P^k] \in G_m(R/P^k) : a_{ii} \in 1 + P^{k-1}, a_{ij} \in P^{k-1}(i, j = 1, \ldots, m, i \neq j)\}$. Hence the order of $Ker(\theta)$ can be computed by $|Ker(\theta)| = (|P^{k-1}/P^k|)^{m^2} = (|P^{k-1}|/|P^k|)^{m^2}$. By (1), the order of $G_m(R/P^k)$ can be computed by $|G_m(R/P^k)| = |Ker(\theta)| \cdot |G_m(R/P^{k-1})| = (|P^{k-1}|/|P^k|)^{m^2} \cdot |G_m(R/P^{k-1})|$ for all positive integer m.

(3) By (2) and mathematical induction on k, we can compute $|G_m(R/P^k)| = (|P/P^k|)^{m^2} \cdot |G_m(R/P)|$. Since R/P is a finite field, by [2 , Theorem VIII.19], $|G_m(R/P)| = (|R/P|^m - 1)(|R/P|^m - |R/P|) \cdots (|R/P|^m - |R/P|^{m-1})$. Hence we have the result. □

Corollary 2.8. *Let p be a prime integer, k and m be any positive integers and $\mathbb{Z}_{p^k}$ be the ring of integers modulo p^k. Then $|G_m(\mathbb{Z}_{p^k})| = p^{m^2} \cdot |G_m(\mathbb{Z}_{p^{k-1}})| = \cdots = p^{(k-1)m^2} \cdot |G_m(\mathbb{Z}_p)|$, where $|G_m(\mathbb{Z}_p)| = (p^m - 1)(p^m - p) \cdots (p^m - p^{m-1})$.*

Proof. Since $\mathbb{Z}_{p^k}$ is a finite local commutative ring and $P = p\mathbb{Z}_{p^k}$ is the unique maximal ideal of $\mathbb{Z}_{p^k}$, we have the result by Theorem 2.7. □

COROLLARY 2.9. *Let m and k be any positive integers. If $p_1^{n_1} \cdot p_2^{n_2} \cdots p_s^{n_s}$ is the prime factorization of k, then the order of $G_m(\mathbb{Z}_k)$ can be computed by $|G_m(\mathbb{Z}_k)| = |G_m(\mathbb{Z}_{p_1^{n_1}})| \cdot |G_m(\mathbb{Z}_{p_2^{n_2}})| \cdots |G_m(\mathbb{Z}_{p_s^{n_s}})|$.*

Proof. It follows from Corollary 1.7 and Corollary 2.8. □

3. The order of $G_m(R)$ when R is a noncommutative ring

Let R be a finite (not necessary commutative) ring and $J(R)$ be the Jacobson radical of R. In this section, we will also compute $|G_m(R)|$, the order of $G_m(R)$, for all positive integers m. By the Wedderburn-Artin Theorem, $M_m(R)/J(M_m(R)) \cong \oplus_{i=1}^n M_i(F_i)$, where $M_i(F_i)$ is the full matrix ring of all n_i by n_i matrices over a finite field F_i for each $i = 1, 2, \ldots, n$ and for some positive integer n_i.

LEMMA 3.1. *Let R be a ring and $G(R)$ be the group of all units in R. Then $G(R)/(1 + J(R)) \cong G(R/J(R))$.*

Proof. Note that the map $\phi : G(R) \to G(R/J(R))$ defined by $\phi(g) = g + J(R)$ for all $g \in G(R)$ is epimorphism and $ker(\phi) = 1 + J(R)$. Hence we have $G(R)/(1+J(R)) \cong G(R/J(R))$ by the First Fundamental Homomorphism Theorem of groups. □

COROLLARY 3.2. *Let R be a finite (not necessary commutative) ring such that $M_m(R)/J(M_m(R)) \cong \oplus_{i=1}^n M_i(F_i)$, where $M_i(F_i)$ is the full matrix ring of all $n_i \times n_i$ matrices over a finite field F_i for each $i = 1, 2, \ldots, n$ and for some positive integer n_i. Then $|G_m(R)| = |J(R)|^{m^2} \cdot \prod_{i=1}^n |G_i(F_i)|$, where $G_i(F_i)$ is the group of all nonsingular matrices in $M_i(F_i)$ for all $i = 1, \ldots, n$.*

Proof. Since $M_m(R)/J(M_m(R)) \cong M_m(R/J(R))$, $M_m(R/J(R)) \cong \oplus_{i=1}^n M_i(F_i)$ and so $G_m(R/J(R)) \cong \oplus_{i=1}^n G_i(F_i)$. Since $J(M_m(R)) = M_m(J(R))$ and $|1 + J(M_m(R))| = |J(M_m(R))|$, by Lemma 3.1 we have $|G_m(R)| = |1+J(M_m(R))| \cdot \prod_{i=1}^n |G_i(F_i)| = |J(M_m(R))| \cdot \prod_{i=1}^n |G_i(F_i)|$ $= |M_m(J(R))| \cdot \prod_{i=1}^n |G_i(F_i)| = |J(R)|^{m^2} \cdot \prod_{i=1}^n |G_i(F_i)|$. □

COROLLARY 3.3. *Let R be a finite (not necessary commutative) local ring. Then $|G_m(R)| = |J(R)|^{m^2} \cdot |G_m(R/J(R))|$.*

Proof. By Lemma 3.1, $G_m(R)/(1+J(M_m(R)) \cong G_m(R/J(R))$. Hence $|G_m(R)| = |J(R)|^{m^2} \cdot |G_m(R/J(R))|$ by the similar argument given in the proof of Corollary 3.2. □

Remark 2. Let R be a finite commutative local ring. Since the unique maximal ideal of R is the Jacobson radical J of R and $J^k = (0)$ for some positive integer k, by Theorem 2.6 $|G_m(R)| = |G_m(R/J^k)| = (|J/J^k|)^{m^2} \cdot |G_m(R/J)| = |J|^{m^2} \cdot |G_m(R/J)|$ for all positive integer m. Even though R is not commutative, $|G_m(R)| = |J|^{m^2} \cdot |G_m(R/J)|$ holds by Corollary 3.3.

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