Lemma 1. *Let $H \leq \mathbb{Q}^\times$ be a finitely generated subgroup. Assume that H does not contain minus a square of $\mathbb{Q}^\times$ or that $m = 1$. Then we have*

$$\left[\mathbb{Q}_{2^m}\left(\sqrt{H}\right) : \mathbb{Q}_{2^m}\right] = \begin{cases} \#\overline{H}/2 & \text{if } m \geq 3 \text{ and } \exists b \in H \text{ with } b \equiv \pm 2 \pmod{\mathbb{Q}^{\times 2}}, \\ \#\overline{H} & \text{otherwise.} \end{cases}$$

where $\overline{H}$ is the image of $H \cdot \mathbb{Q}^{\times 2}$ in $\mathbb{Q}^\times / \mathbb{Q}^{\times 2}$.

Proof. Clearly we may assume that H is generated by suqarefree integers $\{g_1, \ldots, g_r\}$, where $r = \#\overline{H}$. In fact, we have that $\mathbb{Q}_{2^m}(\sqrt{H}) = \mathbb{Q}_{2^m}(\sqrt{H'})$ for any H' such that $(H \cdot \mathbb{Q}^{\times 2})/\mathbb{Q}^{\times 2} = (H' \cdot \mathbb{Q}^{\times 2})/\mathbb{Q}^{\times 2}$. Recall moreover that by Lemma 13 if there is ± 2 times a square in H we can assume that, say, $g_1 = \pm 2$.

Assume first that $m \geq 2$, so that $-1 \notin H$ by assumption. In this case we can work over $\mathbb{Q}_4$ and use Theorem 18 of [1]. We just need to compute the divisibility parameters over $\mathbb{Q}_4$:

$$d_1 = \begin{cases} 0 & \text{if } g_1 \neq \pm 2 \\ 1 & \text{if } g_1 = \pm 2 \end{cases}, \qquad d_i = 0 \quad \text{for } i = 2, \ldots, r,$$

$$h_1 = \begin{cases} 0 & \text{if } 0 \leq g_1 \neq 2 \\ 1 & \text{if } -2 \neq g_1 < 0 \\ 2 & \text{if } g_1 = \pm 2 \end{cases}, \qquad h_i = \begin{cases} 0 & \text{if } g_i > 0 \\ 1 & \text{if } g_i < 0 \end{cases} \quad \text{for } i = 2, \ldots, r.$$

Thus, keeping the notation of the aformentioned Theorem, we get

$$n_1 = \min(1, d_1) = \begin{cases} 0 & \text{if } g_1 \neq \pm 2 \\ 1 & \text{if } g_1 = \pm 2 \end{cases}, \qquad n_i = 0 \quad \text{for } i = 2, \ldots, r.$$

Thus we get

$$\begin{aligned} v_2\left[\mathbb{Q}_{2^m}(\sqrt{H}) : \mathbb{Q}_{2^m}\right] &= \max(h_1 + n_1, \ldots, h_r + n_r, m) - m + r - \sum_{i=1}^r n_i = \\ &= \begin{cases} \max(3, m) - m + r - \sum_{i=1}^r n_i & \text{if } \pm 2 \in H \\ r - \sum_{i=1}^r n_i & \text{if } \pm 2 \notin H \end{cases} \\ &= \begin{cases} 1 + r - 1 & \text{if } m = 2 \text{ and } \pm 2 \in H \\ r - 1 & \text{if } m \geq 3 \text{ and } \pm 2 \in H \\ r & \text{if } \pm 2 \notin H \end{cases} \end{aligned}$$

which is what we want.

Assume now that $m = 1$. If $-1 \notin H$, we get the desired result directly from Lemma 19 of [1] applied with $G = H$, using the computations that we did in the previous case. In case $-1 \in H$, let H' be any subgroup of H such that $H = H' \oplus \langle -1 \rangle$. Notice that we have $\#\overline{H'} = r - 1$, so that Lemma 19 with $G = H'$ again gives our result, and the Proposition is proved. □

Let $G \leq \mathbb{Q}^\times$ be a finitely generated torsion-free subgroup of rank r and let M and n be integers such that $2^n \mid M$. We want to compute the degree

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M : \mathbb{Q}_{2^n}\right]. \tag{1}$$

We will use the same notation as that of Remark 17 of Pietro's file.

1 Case $G \leq \mathbb{Q}_+^\times$

Assume that $G \leq \mathbb{Q}_+^\times$. In this case, by Remark 17, we have that

$$\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M = \mathbb{Q}_{2^n}\left(\sqrt{H}\right).$$

Let $\overline{H}$ be the image of H in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. By Remark 17 and Lemma 1 above, the degree (1) is given by

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right)\cap\mathbb{Q}_M:\mathbb{Q}_{2^n}\right]=\begin{cases}\#\overline{H}/2 & \text{if } n\geq 3 \text{ and } 2\in H,\\ \#\overline{H} & \text{otherwise.}\end{cases}$$

2 General case

Let $\mathcal{B}$ be a basis for G and let $\mathcal{B}_i\subseteq\mathcal{B}$ be the subset of basis elements of 2-divisibility i. Call also $L=\max d_i$ the largest 2-divisiblity parameter. In this way $\mathcal{B}_0,\dots,\mathcal{B}_L$ is a partition of $\mathcal{B}$.

As explained in (ref) we may assume that there is at most one negative basis element. Since we have dealt with the $G\subseteq\mathbb{Q}_+$ case in the previous section, we assume that such an element exists and that it has 2-divisibility d. We call this element g_0.

It is (or will be?) clear (but we should explain it) that it actually does not matter if we have negative elements of divisibility 0: that case is treated exactly as the case $G\subseteq\mathbb{Q}_+$. In conclusion, we assume that:

$$\begin{aligned}\mathcal{B}_1,\dots,\mathcal{B}_{d-1},\mathcal{B}_{d+1},\dots,\mathcal{B}_L&\subseteq\mathbb{Q}_+,\\ g_0<0 \text{ and } \mathcal{B}_d\setminus\{g_0\}&\subseteq\mathbb{Q}_+,\\ d&\geq 1.\end{aligned}$$

We also let

$$N=\begin{cases}\max(3,L) & \text{if } d\neq L,\\ \max(3,L+1) & \text{if } d=L.\end{cases}$$

2.1 General case, $n=1(\leq d)$

This case can be treated as follows: let $\mathcal{S}'=\mathcal{S}\cup\{-1\}$ and let H' be constructed from $\mathcal{S}'$ in the exact same way as H is constructed from $\mathcal{S}$. Then it's easy to check (it follows from the "torsion case" for G, it is for sure in some other file) that $\mathbb{Q}_{2^n}\left(\sqrt{H'}\right)=\mathbb{Q}_{2^{w'}}\left(\sqrt{H}\right)$, where $w'=\min(v_2(M),n+1)$ (as in Remark 17). Then we can again use Lemma 1 and conclude that

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right)\cap\mathbb{Q}_M:\mathbb{Q}_{2^n}\right]=\#\overline{H'},$$

where $\#\overline{H'}$ is the image of H' in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

2.2 General case, $n=2\leq d$

We consider two cases:

- If $v_2(M)=2$ we have $\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right)\cap\mathbb{Q}_M:\mathbb{Q}_{2^n}\right]=\left[\mathbb{Q}_4\left(\sqrt{H}\right):\mathbb{Q}_4\right]=\#\overline{H}$ by Lemma 1.
- If $v_2(M)\geq 3$ we have

$$\begin{aligned}\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right)\cap\mathbb{Q}_M:\mathbb{Q}_{2^n}\right]&=\left[\mathbb{Q}_8\left(\sqrt{H}\right):\mathbb{Q}_4\right]=\left[\mathbb{Q}_8\left(\sqrt{H}\right):\mathbb{Q}_8\right]\cdot[\mathbb{Q}_8:\mathbb{Q}_4]=\\ &=2\left[\mathbb{Q}_8\left(\sqrt{H}\right):\mathbb{Q}_8\right],\end{aligned}$$

  which, by Lemma 1, is given by $\#\overline{H}$ if $2\in H$ and by $2\#\overline{H}$ otherwise.

2.3 General case, $3\leq n\leq d$

We consider two cases:

- If $v_2(M)=3$, by lemma 1 we have

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right)\cap\mathbb{Q}_M:\mathbb{Q}_{2^n}\right]=\left[\mathbb{Q}_8\left(\sqrt{H}\right):\mathbb{Q}_8\right]=\begin{cases}\#\overline{H}/2 & \text{if } \pm 2\in H,\\ \#\overline{H} & \text{otherwise.}\end{cases}$$

- If $v_2(M) \geq 4$ we have

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M : \mathbb{Q}_{2^n}\right] = \left[\mathbb{Q}_{16}\left(\sqrt{H}\right) : \mathbb{Q}_8\right] = \left[\mathbb{Q}_{16}\left(\sqrt{H}\right) : \mathbb{Q}_{16}\right] \cdot \left[\mathbb{Q}_{16} : \mathbb{Q}_8\right] = \\ = 2\left[\mathbb{Q}_{16}\left(\sqrt{H}\right) : \mathbb{Q}_{16}\right],$$

which, by Lemma 1, is given by $\#\overline{H}$ if $2 \in H$ and by $2\#\overline{H}$ otherwise.

2.4 General case, $n \geq d+2$

By the corresponding case in Remark 17, we simply have

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M : \mathbb{Q}_{2^n}\right] = \begin{cases} \#\overline{H'}/2 & \text{if } \pm 2 \in H, \\ \#\overline{H'} & \text{otherwise.} \end{cases}$$

where H' is constructed from $\mathcal{S}' = \mathcal{S} \cup \{B_0\}$ and $\overline{H'}$ is the image of H' in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$.

2.5 General case, $n = d+1$

We distinguish between some cases.

- Assume $n = 2$ (thus $d = 3$) and $v_2(g_0) = 2$ (i.e. 2 divides the square-free part of B_0, where $g_0 = -B_0^{2^d}$). Then we write the square-free part of B_0 as $2s$ for some odd square-free $s \in \mathbb{Z}$. Then letting $\mathcal{S}' := \mathcal{S} \cup \{s\}$ and construct H' from $\mathcal{S}'$ in the usual way. By Remark 17 we have

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M : \mathbb{Q}_{2^n}\right] = \left[\mathbb{Q}_{2^n}\left(\sqrt{H'}\right) : \mathbb{Q}_{2^n}\right] = \#\overline{H'}.$$

But we can be more precise and say that

$$\#\overline{H'} = \begin{cases} 2\#\overline{H} & \text{if } \sqrt{xs} \in \mathbb{Q}_M \text{ for some } x \in \mathcal{S} \text{ and } s \notin \mathcal{S}, \\ \#\overline{H} & \text{otherwise.} \end{cases}$$

- Assume $n \geq 2$ and $2^{n+1} \nmid M$. Then

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M : \mathbb{Q}_{2^n}\right] = \begin{cases} \#\overline{H}/2 & \text{if } \pm 2 \in H \text{ and } n \geq 3, \\ \#\overline{H} & \text{otherwise.} \end{cases}$$

- Assume $n \geq 2$ and $2^{n+1} \mid M$. Following the notation of Remark 17, we have

$$\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M = \mathbb{Q}_{2^n}\left(\sqrt{\langle H, H' \rangle}\right)$$

hence

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M : \mathbb{Q}_{2^n}\right] = \left[\mathbb{Q}_{2^n}\left(\sqrt{\langle H, H' \rangle}\right) : \mathbb{Q}_{2^n}\right] = \\ = \left[\mathbb{Q}_{2^n}\left(\sqrt{\langle H, H' \rangle}\right) : \mathbb{Q}_{2^n}\left(\sqrt{H}\right)\right] \cdot \left[\mathbb{Q}_{2^n}\left(\sqrt{H}\right) : \mathbb{Q}_{2^n}\right].$$

We claim that

$$\left[\mathbb{Q}_{2^n}\left(\sqrt{\langle H, H' \rangle}\right) : \mathbb{Q}_{2^n}\left(\sqrt{H}\right)\right] = \begin{cases} 1 & \text{if } H' = \emptyset \text{ or } H' = \{2\zeta_4\}, \\ 2 & \text{otherwise.} \end{cases}$$

To see this, notice that $\sqrt{2\zeta_4} = \zeta_8\sqrt{2} \in \mathbb{Q}_4 \subseteq \mathbb{Q}_{2^n}\left(\sqrt{H}\right)$, so the first case is settled. Assume now that there is $x = \zeta_{2^n} b \in H'$ with $x \neq 2\zeta_4$. If $y = \zeta_{2^n} c$ is any other element of H', then we have $\sqrt{x/y} = \sqrt{b/c}$. So if $x, y \in \mathbb{Q}_{2^n}\left(\sqrt{\langle H, H' \rangle}\right)$ we have also $\sqrt{b/c} \in \mathbb{Q}_{2^n}\left(\sqrt{\langle H, H' \rangle}\right)$, which by Kummer theory implies $bc \in H$. But then $y \in \mathbb{Q}_{2^n}\left(\sqrt{H}\right)(x)$. So we have $\mathbb{Q}_{2^n}\left(\sqrt{\langle H, H' \rangle}\right) = \mathbb{Q}_{2^n}\left(\sqrt{H}\right)(x)$, and the sought degree is $\left[\mathbb{Q}_{2^n}\left(\sqrt{H}\right)(x) : \mathbb{Q}_{2^n}\left(\sqrt{H}\right)\right]$, which is in fact 2 (Do we need to explain this better?).

We conclude that

$$\left[\mathbb{Q}_{2^n}\left(G^{1/2^n}\right) \cap \mathbb{Q}_M : \mathbb{Q}_{2^n}\right] = \begin{cases} \#\overline{H}/2 & \text{if } n \geq 3,\ \pm 2 \in H \text{ and } H' \subseteq \{2\zeta_4\}, \\ \#\overline{H} & \text{if } (n < 3 \text{ or } \pm 2 \notin H) \text{ and } H' \subseteq \{2\zeta_4\}, \\ \#\overline{H} & \text{if } n \geq 3,\ \pm 2 \in H \text{ and } H' \nsubseteq \{2\zeta_4\}, \\ 2 \cdot \#\overline{H} & \text{if } (n < 3 \text{ or } \pm 2 \notin H) \text{ and } H' \nsubseteq \{2\zeta_4\}. \end{cases}$$

References

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