We begin by giving the code for the function that computes the adelic failure, both in SageMath and in pseudocode. Then we procede to breaking it down into different subcases, in order to check that it computes the correct values.

1 The SageMath Code

The function `adelic_failure_gb` takes two parameters as input: a list $B = \{B_0, \dots, B_t\}$ and an integer d. Each B_i is itself a list of elements of G, and we require the following:

- Each element of $B_i = \{B_{i,0}, \dots, B_{i,t_i}\}$ has 2-divisibility i, using the terminology of [1].
- $\mathcal{B} = \bigcup_{i=1}^{t} B_i$ is a 2-maximal basis for G.
- The integer d is either -1 or $1 \leq d \leq t$. For $i \in \{1, \dots, t\} \setminus \{d\}$ we have $B_i \subseteq \mathbb{Q}_+$. If $d \neq -1$ we have $B_{d,0} < 0$ and $B_{d,j} > 0$ for $j \neq 0$.

The output is a list $A = \{A_1, \dots, A_{N_0}\}$, where each $A_n = \{A_{n,0}, \dots, A_{n,r_n}\}$ is a list of pairs $A_{n,i} = (d_{n,i}, f_{n,i})$. We have $N_0 = \max(3, t+1)$ if $d = t$, while $N_0 = \max(3, t)$ otherwise. For each $1 \leq n \leq N_0$ and each $i \leq r_n$, the integer $d_{n,i}$ is a divisor of $M_0 = d_{N_0, r_{N_0}}$ and a multiple of 2^i, and $f_{n,i}$ is the "adelic failure" (old definition), i.e.:

$$f_{n,i} = \left[\mathbb{Q}_{2^i}\left(G^{1/2^i}\right) \cap \mathbb{Q}_{d_{i,n}} : \mathbb{Q}_{2^i}\right].$$

```
def adelic_failure_gb( B, d ):

    ad_fail = []  # The table to be returned at the end.

    if d == len(B)-1:
        N = max(3,len(B)+1)
    else:
        N = max(3,len(B))

    # The shortlist grows at each step, so we build it incrementally.
    shortlist = []
    # The "special element" is (n,b) = \zeta_{2^n}\sqrt{b}.
    special_element = (1,1)

   M = 1  # M also grows with n.

    for n in range( 1, N+1 ): # Read as: 1 \leq n \leq N

        # We add the new elements to the shortlist, modifying M if needed.
        # This is not done in case we are in the extra "fake" level.
        if n-1 < len(B):
            for g in B[n-1]:
                if g < 0 and n > 1:
                    special_element = ( n+1, abs(g)^(1/(2^(n-1))) )
                   M = lcm( M, special_embed( special_element ) )
                else:
                    b = g^(1/(2^(n-1))) # b is 2-indivisible
                    shortlist.append( b )
                   M = lcm( M, cyc_embed(b) )

        # We add a root of an even power of the negative generator, as soon as
        # we are beyond its level.
        if d != -1 and n == d+2:
```

```
            b = abs(B[d][0])^(1/2^d)
            shortlist.append( b )
            M = lcm( M,  cyc_embed(b) )

        M = lcm(M,2^n)

        if n <= d:
            M = lcm( M,  2^(n+1) )

        if n == 1 and d >= 1:
            shortlist.append(-1)
        if n > 1 and -1 in shortlist:
            shortlist.remove(-1)

        aux = [] # Next line of ad_fail table

        for dM in divisors( M ):
            if dM % (2^n) != 0:
                continue

            S = [ product(s) for s in subsets( shortlist ) ]
            H = [ cyc_embed( s ) for s in S ]
            r = len( [ b for b in H if dM % b == 0 ] )

            if n <= d and dM % (2^(n+1)) == 0 and n > 1:
                r *= 2

            if 8 in H and dM % 8 == 0 and (n >= 3 or (n == 2 and n <= d)):
                r = r/2

            if special_element != (1,1) and special_element[0] == n+1:
                nothing_to_do = False
                intersecting_QdM = False
                for s in S:
                    new_special = ( n+1, special_element[1] * s )
                    m = special_embed( new_special )
                    if n == 2 and m == 4: # \zeta_8 times 2 times square
                        nothing_to_do = True
                    if dM % m == 0:
                        intersecting_QdM = True
                if intersecting_QdM and not nothing_to_do:
                    r *= 2

            aux.append( (dM,r) )

        ad_fail.append(aux)

    return ad_fail
```

We have used the following auxiliary functions:

```
# Computes the minimal cyclotomic field containing \sqrt(b)
def cyc_embed( b ):
    m = squarefree_part(b)
    if m%4 != 1:
        m *= 4
    return abs(m)
```

```
# Computes the minimal cyclotomic field containing \zeta_{2^n}\sqrt(b)
def special_embed( (n,b) ):
    m = squarefree_part(b)
    if n == 3 and m % 2 == 0:
        return 4 * cyc_embed(m/2)
    else:
        return lcm( 2^n, cyc_embed(b) )
```

2 The Pseudocode

We translate the SageMath code into pseudocode for ease of readability.

Algorithm 1 Compute the adelic failure

```
Let B, t, d and N as described in the previous section
Let M ← 1, special_element ← 1 and shortlist ← []

for n = 1 to N do
    if n − 1 < t then
        for g ∈ B_{n−1} do
            if g < 0 and n > 1 then
                special_element ← (n + 1, (2^{n−1})√|g|)
                M ← lcm(M, special_embed(special_element))
            else
                Add (2^{n−1})√g to shortlist
                M ← lcm(M, cyc_embed(g))
            end if
        end for
    end if

    if n = d + 2 and d ≠ −1 then
        Add (2^d)√|B_{d,0}| to shortlist
        M ← lcm(M, cyc_embed(|B_{d,0}|))
    end if

    if n ≤ d then
        M ← lcm(M, 2^{n+1})
    else
        M ← lcm(M, 2^n)
    end if

    if n = 1 and d ≥ 1 then
        Add −1 to shortlist
    end if

    if n > 1 then
        Remove −1 from shortlist (if present)
    end if
```

for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ **do**
$S \leftarrow \left\{\prod_{x\in T} x \,|\, T \subseteq \texttt{shortlist}\right\}$
$H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \,|\, \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
$r \leftarrow \#\{s \in S \,|\, \sqrt{s} \in \mathbb{Q}_{d_M}\}$

if $q < n \leq d$ and $2^{n+1} \mid d_M$ **then**
$r \leftarrow 2r$
end if

if $8 \in H$ and $8 \mid d_M$ and (either $n \geq 3$ or $n = 2 \leq d$) **then**
$r \leftarrow r/2$
end if

if $\texttt{special\_element} = \zeta_{2^{n+1}}\sqrt{b}$ for some $b \in \mathbb{Q}$ **then**
$\texttt{specials} \leftarrow \{\zeta_{2^{n+1}}\sqrt{bs} \,|\, s \in S\}$
if $\exists x \in \texttt{specials}$ such that $x \in \mathbb{Q}_{d_M}$ and $\texttt{special\_embed}(s) \neq 4 \,\forall s \in \texttt{specials}$ **then**
$r \leftarrow 2r$
end if
end if

Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = r.$
end for
end for

3 Pseudocode, the sub-cases

We divide the pseudocode in sub-cases.

3.1 Case $G \leq \mathbb{Q}_+^\times$

Algorithm 2 Adelic failure, case $G \leq \mathbb{Q}^\times$

for $n = 1$ to N **do**
for $g \in B_{n-1}$ **do**
Add $\sqrt[2^{n-1}]{g}$ to $\texttt{shortlist}$
$M \leftarrow \operatorname{lcm}(M, \texttt{cyc\_embed}(g))$
end for

$M \leftarrow \operatorname{lcm}(M, 2^n)$

for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ **do**
$S \leftarrow \left\{\prod_{x\in T} x \,|\, T \subseteq \texttt{shortlist}\right\}$
$H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \,|\, \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
$r \leftarrow \#\{s \in S \,|\, \sqrt{s} \in \mathbb{Q}_{d_M}\}$
Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H \text{ and } n \geq 3, \\ r & \text{otherwise.} \end{cases}$
end for
end for

3.2 Case $d \neq -1$, $n \leq d$

For this and the following cases, we assume we are already inside the main `for` cycle, since we have particular assumptions on n.

Algorithm 3 Adelic failure, case $d \neq -1$, $n \leq d$

---

for $g \in B_{n-1}$ **do**
    Add $\sqrt[2^{n-1}]{g}$ to `shortlist`
    $M \leftarrow \mathrm{lcm}(M, \texttt{cyc\_embed}(g))$
end for

$M \leftarrow \mathrm{lcm}(M, 2^{n+1})$

if $n = 1$ and $d \geq 1$ **then**
    Add -1 to `shortlist`
end if

if $n > 1$ **then**
    Remove -1 from `shortlist` (if present)
end if

for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ **do**
    $S \leftarrow \left\{\prod_{x \in T} x \mid T \subseteq \texttt{shortlist}\right\}$
    $H \leftarrow \{\min \{x \in \mathbb{Z}_{>0} \mid \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
    $r \leftarrow \# \{s \in S \mid \sqrt{s} \in \mathbb{Q}_{d_M}\}$

    if $n > 1$ and $2^{n+1} \mid d_M$ **then**
        $r \leftarrow 2r$
    end if
    Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H \text{ and } n \geq 3, \\ r/2 & \text{if } 8 \in H \text{ and } n = 2 \text{ and } 8 \mid d_M \\ r & \text{otherwise.} \end{cases}$
end for

3.3 Case $d \neq -1$, $n \geq d+2$

Algorithm 4 Adelic failure, case $d \neq -1$, $n \geq d+2$

if $n-1 < t$ **then**
  for $g \in B_{n-1}$ **do**
    Add $\sqrt[2^{n-1}]{g}$ to `shortlist`
    $M \leftarrow \operatorname{lcm}(M, \texttt{cyc\_embed}(g))$
  end for
end if

if $n = d+2$ **then**
  Add $\sqrt[2^d]{|B_{d,0}|}$ to `shortlist`
  $M \leftarrow \operatorname{lcm}(M, \texttt{cyc\_embed}(|B_{d,0}|))$
end if

$M \leftarrow \operatorname{lcm}(M, 2^n)$

for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ **do**
  $S \leftarrow \left\{\prod_{x \in T} x \mid T \subseteq \texttt{shortlist}\right\}$
  $H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \mid \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
  $r \leftarrow \#\{s \in S \mid \sqrt{s} \in \mathbb{Q}_{d_M}\}$
  Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H, \\ r & \text{otherwise.} \end{cases}$
end for

3.4 Case $d \neq -1$, $n = d + 1$

Algorithm 5 Adelic failure, case $d \neq -1$, $n = d + 1$

for $g \in B_{n-1}$ **do**
  if $g < 0$ **then**
    `special_element` $\leftarrow (n+1, \sqrt[2^{n-1}]{|g|})$
    $M \leftarrow \mathrm{lcm}(M, \texttt{special\_embed}(\texttt{special\_element}))$
  else
    Add $\sqrt[2^{n-1}]{g}$ to `shortlist`
    $M \leftarrow \mathrm{lcm}(M, \texttt{cyc\_embed}(g))$
  end if
end for

$M \leftarrow \mathrm{lcm}(M, 2^n)$

Remove -1 from `shortlist` (if present)

for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ **do**
  $S \leftarrow \left\{\prod_{x \in T} x \,|\, T \subseteq \texttt{shortlist}\right\}$
  $H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \,|\, \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
  $r \leftarrow \#\{s \in S \,|\, \sqrt{s} \in \mathbb{Q}_{d_M}\}$

  if $\exists x \in \{\zeta_{2^{n+1}}\sqrt{bs} \,|\, s \in S\} \cap \mathbb{Q}_{d_M}$ and $\texttt{special\_embed}(s) \neq 4 \, \forall s \in \texttt{specials}$ **then**
    $r \leftarrow 2r$
  end if
  Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H \text{ and } n \geq 3, \\ r & \text{otherwise.} \end{cases}$
end for

References

[1] Debry, C. - Perucca, A.: *Reductions of algebraic integers*, J. Number Theory, **167** (2016), 259–283.