Computation of the adelic failure

The aim of this document is bridge the gap between the theory developed in [2] and the function `adelic_failure_gb` that computes the adelic failure. By reading [2], the pseudo-code in this file and the (commented) SageMath code, one can verify that the script produces the correct results.

We begin by giving the code for the function that computes the adelic failure, both in SageMath and in pseudocode. Then we procede to breaking it down into different subcases.

The SageMath code

The function `adelic_failure_gb` takes two parameters as input: a list $B = \{B_0, \dots, B_t\}$ and an integer d. Each B_i is itself a list of elements of G, and we require the following:

- Each element of $B_i = \{B_{i,0}, \dots, B_{i,t_i}\}$ has 2-divisibility i, using the terminology of [1].
- $\mathcal{B} = \bigcup_{i=1}^{t} B_i$ is a 2-maximal basis for G.
- The integer d is either -1 or $1 \leq d \leq t$. For $i \in \{1, \dots, t\} \setminus \{d\}$ we have $B_i \subseteq \mathbb{Q}_+$. If $d \neq -1$ we have $B_{d,0} < 0$ and $B_{d,j} > 0$ for $j \neq 0$.

The output is a list $A = \{A_n \mid n \text{ divides } N_0\}$, indexed by the positive divisors of N_0, where $N_0 = \max(3, t+1)$ if $d = t$, while $N_0 = \max(3, t)$ otherwise. Each $A_n = \{A_{n,0}, \dots, A_{n,r_n}\}$ is a list of pairs $A_{n,i} = (d_{n,i}, f_{n,i})$. For each $n \mid N_0$ and each $i \leq r_n$, the integer $d_{n,i}$ is a divisor of $M_0 = d_{N_0, r_{N_0}}$ and a multiple of 2^i, and $f_{n,i}$ is the *adelic failure*, that is the degree

$$f_{n,i} = \left[\mathbb{Q}_{2^i}\left(G^{1/2^i}\right) \cap \mathbb{Q}_{d_{n,i}} : \mathbb{Q}_{2^i} \right].$$

The pseudo-code

We translate the SageMath code into pseudocode for ease of readability.

Algorithm 1 Compute the adelic failure

```
Let $B$, $t$, $d$ and $N$ as described in the previous section
Let $M \leftarrow 1$, special_element $\leftarrow 1$ and shortlist $\leftarrow [\,]$

for $n = 1$ to $N$ do
    if $n - 1 < t$ then
        for $g \in B_{n-1}$ do
            if $g < 0$ and $n > 1$ then
                special_element $\leftarrow (n + 1, \sqrt[2^{n-1}]{|g|})$
                $M \leftarrow \mathrm{lcm}(M,$ special_embed(special_element)$)$
            else
                Add $\sqrt[2^{n-1}]{g}$ to shortlist
                $M \leftarrow \mathrm{lcm}(M,$ cyc_embed($g$)$)$
            end if
        end for
    end if

    if $n = d + 2$ and $d \neq -1$ then
        Add $\sqrt[2^d]{|B_{d,0}|}$ to shortlist
        $M \leftarrow \mathrm{lcm}(M,$ cyc_embed($|B_{d,0}|$)$)$
    end if

    if $n \leq d$ then
        $M \leftarrow \mathrm{lcm}(M, 2^{n+1})$
    else
        $M \leftarrow \mathrm{lcm}(M, 2^n)$
    end if

    if $n = 1$ and $d \geq 1$ then
        Add $-1$ to shortlist
    end if

    if $n > 1$ then
        Remove $-1$ from shortlist (if present)
    end if
```

```
    for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ do
        $S \leftarrow \left\{\prod_{x\in T} x \,|\, T \subseteq \texttt{shortlist}\right\}$
        $H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \,|\, \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
        $r \leftarrow \#\{s \in S \,|\, \sqrt{s} \in \mathbb{Q}_{d_M}\}$

        if $q < n \leq d$ and $2^{n+1} \mid d_M$ then
            $r \leftarrow 2r$
        end if

        if $8 \in H$ and $8 \mid d_M$ and (either $n \geq 3$ or $n = 2 \leq d$) then
            $r \leftarrow r/2$
        end if

        if $\texttt{special\_element} = \zeta_{2^{n+1}}\sqrt{b}$ for some $b \in \mathbb{Q}$ then
            $\texttt{specials} \leftarrow \{\zeta_{2^{n+1}}\sqrt{bs} \,|\, s \in S\}$
            if $\exists x \in \texttt{specials}$ such that $x \in \mathbb{Q}_{d_M}$ and $\texttt{special\_embed}(s) \neq 4\,\forall s \in \texttt{specials}$ then
                $r \leftarrow 2r$
            end if
        end if

        Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = r.$
    end for
end for
```

Pseudo-code, the sub-cases

We divide the pseudocode in sub-cases. In each sub-case we apply the trivial simplifications to the pseudo-code above.

Case $G \leq \mathbb{Q}_+^\times$

Algorithm 2 Adelic failure, case $G \leq \mathbb{Q}^\times$

```
for $n = 1$ to $N$ do
    for $g \in B_{n-1}$ do
        Add $\sqrt[2^{n-1}]{g}$ to shortlist
        $M \leftarrow \mathrm{lcm}(M, \texttt{cyc\_embed}(g))$
    end for

    $M \leftarrow \mathrm{lcm}(M, 2^n)$

    for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ do
        $S \leftarrow \left\{\prod_{x \in T} x \,|\, T \subseteq \texttt{shortlist}\right\}$
        $H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \,|\, \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
        $r \leftarrow \#\{s \in S \,|\, \sqrt{s} \in \mathbb{Q}_{d_M}\}$
        Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H \text{ and } n \geq 3, \\ r & \text{otherwise.} \end{cases}$
    end for
end for
```

Case $d \neq -1$, $n \leq d$

For this and the following cases, we assume we are already inside the main `for` cycle, since we have particular assumptions on n.

Algorithm 3 Adelic failure, case $d \neq -1$, $n \leq d$

```
for $g \in B_{n-1}$ do
    Add $\sqrt[2^{n-1}]{g}$ to shortlist
    $M \leftarrow \mathrm{lcm}(M, \texttt{cyc\_embed}(g))$
end for

$M \leftarrow \mathrm{lcm}(M, 2^{n+1})$

if $n = 1$ and $d \geq 1$ then
    Add $-1$ to shortlist
end if

if $n > 1$ then
    Remove $-1$ from shortlist (if present)
end if

for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ do
    $S \leftarrow \left\{\prod_{x \in T} x \mid T \subseteq \texttt{shortlist}\right\}$
    $H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \mid \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
    $r \leftarrow \#\{s \in S \mid \sqrt{s} \in \mathbb{Q}_{d_M}\}$

    if $n > 1$ and $2^{n+1} \mid d_M$ then
        $r \leftarrow 2r$
    end if
    Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H \text{ and } n \geq 3, \\ r/2 & \text{if } 8 \in H \text{ and } n = 2 \text{ and } 8 \mid d_M \\ r & \text{otherwise.} \end{cases}$
end for
```

Case $d \neq -1$, $n \geq d+2$

Algorithm 4 Adelic failure, case $d \neq -1$, $n \geq d+2$

```
if $n - 1 < t$ then
    for $g \in B_{n-1}$ do
        Add $\sqrt[2^{n-1}]{g}$ to shortlist
        $M \leftarrow \mathrm{lcm}(M, \texttt{cyc\_embed}(g))$
    end for
end if

if $n = d + 2$ then
    Add $\sqrt[2^d]{|B_{d,0}|}$ to shortlist
    $M \leftarrow \mathrm{lcm}(M, \texttt{cyc\_embed}(|B_{d,0}|))$
end if

$M \leftarrow \mathrm{lcm}(M, 2^n)$

for all $d_M \in \texttt{divisors}(M)$ such that $2^n \mid M$ do
    $S \leftarrow \left\{\prod_{x \in T} x \mid T \subseteq \texttt{shortlist}\right\}$
    $H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \mid \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
    $r \leftarrow \#\{s \in S \mid \sqrt{s} \in \mathbb{Q}_{d_M}\}$
    Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H, \\ r & \text{otherwise.} \end{cases}$
end for
```

Case $d \neq -1$, $n = d + 1$

Algorithm 5 Adelic failure, case $d \neq -1$, $n = d + 1$

```
for $g \in B_{n-1}$ do
    if $g < 0$ then
        special_element ← $(n+1,\ \sqrt[2^{n-1}]{|g|})$
        $M \leftarrow \mathrm{lcm}(M,$ special_embed(special_element)$)$
    else
        Add $\sqrt[2^{n-1}]{g}$ to shortlist
        $M \leftarrow \mathrm{lcm}(M,$ cyc_embed$(g))$
    end if
end for

$M \leftarrow \mathrm{lcm}(M, 2^n)$

Remove $-1$ from shortlist (if present)

for all $d_M \in$ divisors$(M)$ such that $2^n \mid M$ do
    $S \leftarrow \left\{\prod_{x \in T} x \mid T \subseteq \texttt{shortlist}\right\}$
    $H \leftarrow \{\min\{x \in \mathbb{Z}_{>0} \mid \sqrt{s} \in \mathbb{Q}_x\} \mid s \in S\}$
    $r \leftarrow \#\{s \in S \mid \sqrt{s} \in \mathbb{Q}_{d_M}\}$

    if $\exists x \in \{\zeta_{2^{n+1}}\sqrt{bs} \mid s \in S\} \cap \mathbb{Q}_{d_M}$ and special_embed$(s) \neq 4\ \forall s \in$ specials then
        $r \leftarrow 2r$
    end if
    Declare $\left[\mathbb{Q}_{2^n}\left(\sqrt[2^n]{G}\right) \cap \mathbb{Q}_{d_M} : \mathbb{Q}_{2^n}\right] = \begin{cases} r/2 & \text{if } 8 \in H \text{ and } n \geq 3, \\ r & \text{otherwise.} \end{cases}$
end for
```

Bibliography

[1] Debry, C. - Perucca, A.: *Reductions of algebraic integers*, J. Number Theory, **167** (2016), 259–283.

[2] Perucca, A. - Sgobba, P. - Tronto, S.: *Explicit Kummer Theory for the rational numbers*, preprint.